ELEMENTARY PURE GEOMETRY

WITH

MENSURATION

BY

E. BUDDEN

Elementary Pure Geometry

With Mensuration

A COMPLETE COURSE OF GEOMETRY FOR SCHOOLS

BY

E. BUDDEN, M.A. OXON., B.Sc. LOND.
MACCLESFIELD GRAMMAR SCHOOL;
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PREFACE.

In Parts I. and II. of this book, which cover the whole of Eucl. I.–VI. and the geometrical part of Trigonometry—i.e. to solution of triangles—I have followed generally the Cambridge Previous Syllabus and the recommendations of the Mathematical Association; but much additional matter has been introduced, with modern methods suitable to this stage of the subject.

Part I. consists of experimental geometry, angle and parallels, symmetrical and congruent figures, elementary areas—i.e. all Eucl. I.—decimal measurement, similar and similarly situated figures (linear properties, Eucl. VI. 2–18), and a short account of simple loci.

Part II. contains circle properties (Eucl. III. and IV.), with centre of similitude, radical axis, and tangent circles; areas of parts of divided lines and of similar figures (Eucl. II., VI. 1, 19...); the methods of multiplication (similitude) and rotation, maxima and minima, envelopes, and loci; and it has a chapter on Trigonometry, covering the whole ground to solution of triangles.

Part III. contains an account of modern projective geometry in the plane, elementary geometrical conics, and solid geometry, including the mensuration of cylinder, pyramid, cone, and sphere.

I have sought to make improvements in the following directions:

1. Instruments are used from the very beginning, and parallels, perpendiculars, circles, triangles, &c. drawn and some of their properties arrived at and stated without any formal proof whatever.

2. Experimental geometry leads up to the definitions of the plane, straight line, angle, perpendicular, and direction. These, with definitions of figures and the experimental treatment of the area of the parallelogram, constitute the Introduction and Chapter I.

3. The order has been so arranged that related properties are brought together, and in most cases are dealt with on the same page. Thus the four cases of congruent triangles are taken consecutively in two pages and by one method—viz. direct superposition. The parallelogram and similar triangles immediately follow these.

4. The general plan has been arranged upon the fundamental principle that symmetry precedes congruence. The properties of the isosceles triangle, and the complete cases of congruent triangles, cannot be established until it is shown that an angle is reversible —i.e. that the two faces of a plane angle are congruent. By proving this we can establish the elementary properties of the triangle and circle (Chapter II.) independently, and so gain a good knowledge of the triangle before the more difficult case of two triangles is approached. The failure to prove this invalidates the proofs of most of the fundamental theorems in the ordinary text-books.

It was the proof of this reversibility of the angle that led me to my definition of the plane, and with it, as I had always anticipated, to that of the straight line also. Laplace's definition of a plane, though sufficient as a test (and so used by the great engineer Whitworth), does not give the straight line as the intersection of two planes.

My double treatment of ratio in Chapters III., V., and VII. deserves a special notice. The mensuration of figures requires the numerical treatment of ratio, which is, moreover, easier to understand than the purely geometrical treatment; and for this to be formally rigorous our definition of ratio should be number or measure, and should include irrational numbers, since without these such expressions as sin 41°, considered as numbers, are unintelligible. I have treated this part of the work completely but simply by the method of decimal scales, a sufficient explanation being given in two short notes in Chapters III. and V.

Pure geometry must be independent of the theory of number; and at the end of Chapter VII., in order to complete the account of descriptive geometry, as distinct from that which is partly numerical, I have given a purely geometrical definition and treatment of ratio. My proofs of the propositions of Eucl. II. are also purely geometrical—i.e. *not* derived from mensuration.

Thus, by substituting for the numerical definition of ratio in Chapter III. the geometrical one of Chapter VII., the book gives a complete course of strictly pure elementary geometry.

The whole of this treatment of ratio is, so far as I know, original; though the numerical ratio is only a modification of Dedekind's Schnitt, his complete system of fractions being replaced by terminating decimals.

The use of the decimal scale, however, makes the process much simpler, and as a matter of fact I arrived at my result by a quite different route from that which Dedekind follows. Students interested in Euclid's method should consult Professor M. J. M. Hill's Euclid, V. and VI.

I have borrowed the description of multiplication and rotation from Pedersen's *Méthodes et Théories;* and my knowledge of modern geometry is derived chiefly from Mulcahy, Townsend, Chasles, and Cremona. Students may consult Russell's *Projective Geometry.* The derivation of pole and polar from Pascal's theorem, the form of proof that a conic is a perspective of a circle, and of the converse, and the always real construction for throwing any five points on to a circle, are my own.

I have kept to the focus and directrix definition of a conic because it is so much easier to derive the form and simpler properties of the curve in this way. My construction of the conic from what I have called the focircle, leads quite naturally to the modern treatment of the curve. And further, the focal properties of the conic can be studied simultaneously with the beginning of Chapter VII., so that Chapters VII. and VIII. can be taken concurrently. I hope that these chapters may help to bring modern geometry within the range of the higher mathematical classes of our schools.

I have not hesitated to modify Euclid's proofs, constructions, and order in the interest of simplicity; nor to introduce new symbols and new names where found desirable. The latter I have tried to make so that their meaning is obvious when once learnt—e.g. right bisector, mean part of a line. I have used the symbol $|||$ for 'is similar to,' to suggest the connection of similar figures with parallelism; and as it is also the symbol $\equiv$ for congruence turned up, and very easy for boys to write (much easier,

for instance, than $\simeq$), it seems the obvious one. Geometrical symbols are used for verbs ($\|$ 'is parallel to,' &c.), literal abbreviations (which can be coined as desired) for nouns or adjectives.

Numerous examples are given. Those at the end of each chapter are carefully graded (with here and there a difficult one), and should be taken concurrently with the text. General examples for revision are given at the end of Chapter V.

Answers are given to most of the practical questions in Part I. to serve as checks on the drawing. In the other parts—e.g. in tangency of circles—the drawings are easily checked without numerical aid.

Much care has been taken by the publishers in setting the type and reproducing the figures. The important parts of a figure given in the statement, upon which a proof or construction depends, are represented by thickened lines or points. Mere lines of construction or aids to proof are, in general, dotted; and the finally constructed figure is shown by a thin continuous line.

Important statements (theorems, corollaries, constructions, and a few worked examples) have been set throughout in heavier type; and in the definitions the *thing defined* is distinguished by heavier type.

Short notes and exercises are set in smaller type; notes generally are meant primarily for the teacher, who will exercise his discretion in using or leaving them.

The general plan of headlines; the heavy, medium, and small type; and the thickened, dotted, and thin lines, make the book very easy to use for purpose of reference—the proper function of a text-book.

My thanks are due to my friends, Mr A. E. Holme of Dewsbury (who communicated to me the construction for a perpendicular by set-square), Mr R. W. Batho of St Paul's, Mr G. H. Hughes of Marlborough, and Mr N. Baron of Macclesfield, for assistance with proofs and examples; and to the Committees of the British and Mathematical Associations for making it possible to write a text-book of geometry on a definite plan.

E. BUDDEN.

MACCLESFIELD, 1904.

SUGGESTIONS FOR THE TEACHER.

1. Complete lines are used for the more important parts of a figure, broken lines or parts of lines—e.g. arcs—for the less important. Pupils should be encouraged to adopt this distinction.

2. The instructions and experimental work in the Introduction and in Chapter I. on the use of instruments, on the experimental derivation of plane, straight line, bisectors, perpendiculars, and areas, and the notes in Chapters III., V. on ratio should be taken *orally* with beginners, at the teacher's discretion.

3. The constructions in the Introduction as far as a simple plain scale of inches or centimetres, and the simpler constructions at the end of Chapter II., should be mastered without any formal proof, before the study of formal geometry is attempted.

4. Definitions, and all statements of theorems, should be learnt by heart, as soon as reached in the ordinary course of the book. The examples accompanying these, or to be found at the end of the various chapters, should be taken *pari passu.* They have been carefully graded, though here and there, for convenience of reference, an example from a later stage has been introduced before its true place. Congruent triangles should not be used where proof by symmetry is simpler.

5. The constructions in any chapter should in general be taken in advance of the theorems—their formal proof may be taken as soon as sufficient theory has been mastered. It is hoped that the hints given in the Introduction and in Chapters III., IV., and V. may be helpful in the solution of the more difficult problems.

6. An accuracy of 1 to 2 per cent. is all that can be expected without the conveniences and exact instruments of a drawing-office. $1\frac{3}{4}''$ should be taken as correct for $\sqrt{3}''$.

7. Four-figure tables should be used in Chapter VI.; their use should be briefly explained orally.

CONTENTS.

PART I.

PART II.

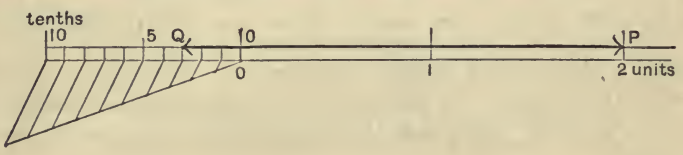
PART III.

PART I.

INTRODUCTION.

INSTRUMENTS—EXPERIMENTAL GEOMETRY—SCALES.

The pupil should be provided with pencil, pencil compass for drawing circles, divider for setting off lengths, straight-edge for ruling straight lines, inch and centimetre scales divided into tenths, inch divided into eighths, and preferably also into hundredths by diagonal division. (The pencil used either for the compass or for ruling straight lines should be hard—e.g. HHH—and its end cut like that of a table-knife. This furnishes an excellent drawing point.) A protractor for drawing angles, and set-squares * for drawing parallels and perpendiculars, are also required.



Lengths are measured by means of a scale. This is a straight-edge divided into suitable units (inch, centimetre, &c.), with at least one unit subdivided into suitable parts, tenths, eighths, &c., as required.

'Measure or set off a length by a scale.'

If PQ is a length to be measured, adjust the **divider** points (not the compass) to its ends, put the divider on the scale with one end on a unit division, say 2, and the other in the divided unit, say at 3. Then if this is divided into tenths,

$$PQ = 2{\cdot}3 \text{ units.}$$

Similarly, to set off a length 2·3 units, set one end of the divider on the 2nd unit, and the other on the 3rd tenth, and prick off the length PQ in any required position.

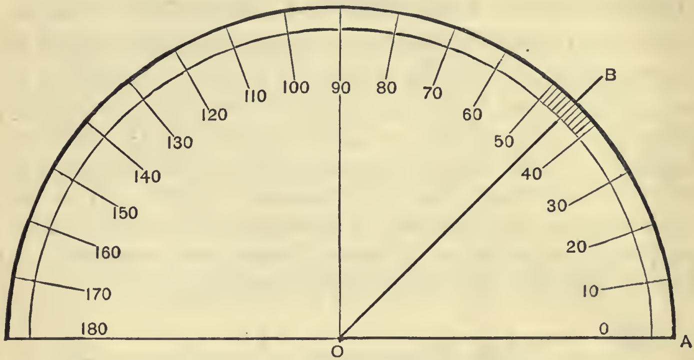
Ex. 1. Set off straight lines of 3·2″,† 2·5 cm., $1\frac{1}{4}$″, $3\frac{1}{2}$ cm.

Ex. 2. Draw circles, radii $1\frac{1}{2}$″, 3·3 cm., 2·2″.

* Two are generally found sufficient. † ″ means inch or inches.

MEASURE OF ANGLES.

The protractor is primarily a semicircle divided into 180 equal parts; the angle at the centre **O** formed by radii of one of these parts is a degree (°), so that the angular space round a point **O**



(using a whole circle) contains 360 degrees. An angle of 90° is called a right angle. In the figure the semicircle is shown divided into arcs of 10° each, and one of them (40°–50°) divided into degrees.

'Measure or construct an angle by protractor.'

If **AOB** is an angle to be measured, place the centre of the protractor at **O**, and the base along **OA**; note the division on the circle which coincides with **OB**. In the figure the division is 45, and angle **AOB** is 45°.

To construct an angle of 45° at a point **O**, set the protractor with its centre at **O** and its base along one side **OA** of the angle, mark with divider a point opposite the 45 division, and rule the straight line **OB** through this point with a straight-edge. Then angle **AOB** is 45°.

Ex. Draw a straight line **AB**, 2″ long, at **A** make angles of 30°, 48°, 57°, 90°, 108°, 156° from **AB**.

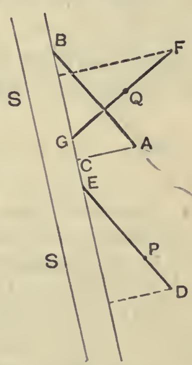
The rectangular form of protractor is easily understood from the semicircular form.

The use of the scale of chords for constructing angles will be found after Construction 4, Chapter II.

SET–SQUARE—PARALLEL—PERPENDICULAR.

Set-squares are right-angled triangles, used with a straight-edge for drawing parallels, perpendiculars, (and one or two special angles).

'Construct a parallel to a straight line from a given point.'



Draw a straight line AB, and mark a point P.

Put one edge of the set-square along AB, bring a straight-edge SS along another side BC of the set-square, and hold the straight-edge quite firm with the left hand.

Slide the set-square along the straight-edge until the first edge AB traverses the point P, in the position DE.

Hold the set-square firm, and rule the line PE.

PE is parallel to AB.

Note. This can be proved as soon as Def. 15, Chapter I., is reached.

'Construct a perpendicular to a straight line from a given point.'

Draw a straight line AB, and mark a point Q (above figure).

Put the **hypotenuse** (longest side) of the set-square along AB, bring a straight-edge SS along another side BC of the set-square, and hold the straight-edge quite firm.

Now turn the set-square * so that its third side AC is along the straight-edge, and slide it along until the hypotenuse traverses Q in the position FG.

Hold the set-square firm, and rule the line QG.

QG is perpendicular to AB.

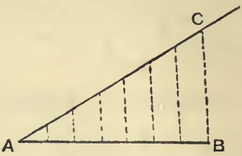
Note. This can be proved by Theorem 14, Chapter II.

Ex. Draw a straight line AB, and draw AC, making angle BAC 70°. Draw BD parallel to AC and measure angle ABD. Also draw BE perpendicular to AB.

* The set-square must be turned *round*, not turned over.

Division of Lines.

The set-square is also used to divide a straight line into any number of equal parts by drawing parallels.



Thus, if **AB** in the figure is to be divided into 7 equal parts:

Draw another line **AC**, and with compass or divider set off 7 equal parts along it as far as **C**, say.

Draw parallels to **BC** through the points of division; these divide **AB** into 7 equal parts. (Theorem 34, Chapter III.)

EXAMPLES—I.

1. Draw a straight line 10 cm. long and measure it in inches. How many inches in a centimetre? Centimetres in an inch?

2. Make a straight line $1\frac{1}{4}''$ long. Make angles of 70° and 50° at its ends, forming a triangle. Measure the third angle.

3. Make a straight line **AB**, 5 cm.; with centres **A**, **B**, radius 5 cm., draw two circles meeting in **C**; join **AC**, **BC**. What do you know about the straight lines **AC**, **BC**, **AB**?

4. Measure the angle **ACB** in Ex. 3.

5. Make a straight line **AB**, $1\frac{1}{2}''$; make **AC** perpendicular to it, $2''$; measure **BC**.

6. If a straight line **AB** points east, draw straight lines **AC**, **AD**, **AE** pointing N.E., S.W., and S. Are any two of these parts of one straight line?

7. In question 6, what is the number of degrees in the angles **BAC**, **BAD**, **BAE**?

8. Draw a straight line **AB**, $1\frac{3}{4}''$; at **A** make an angle **BAC**, 40°. Take a point **D** in **AB**, $1\frac{1}{4}''$ from **A**; draw **DE** parallel to **AC**.

9. Measure the angle **BDE** in Ex. 8. Measure also **ADE**.

10. Draw a straight line $2\frac{1}{2}''$; divide it into 5 equal parts.

11. Draw a straight line **OA**, $6''$. From **O** mark off successive lengths of $1''$; divide the first inch into quarters.

12. Using the scale of Ex. 11, set off a length of $2\frac{3}{4}''$, and draw a circle with this radius.

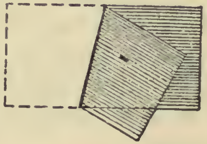
13. Bisect (by parallels) a line 5 cm. long, and draw a perpendicular to it through the mid point.

14. Take a point **A** on the perpendicular of Ex. 13, 3 cm. from one end of the first line, and measure its distance from the other end.

EXPERIMENTAL GEOMETRY.

The Straight Line.

Fold a sheet of fairly stiff paper, and mark the fold as accurately as you can.



Using the fold as guide, rule a pencil line on paper. Place your straight-edge along the line. Is the line straight?

Set your fold along the straight-edge. Is the fold straight?

If a fold is made in this manner, and the folded sheet pressed out on a flat surface, the fold is practically straight. A very good ruler can be extemporised by twice folding a sheet of foolscap.

Set the edges of two set-squares together; can you see daylight between them? If you can, the edges cannot both be straight.

Rule a line, using a straight-edge with the edge on the right of the instrument; turn the instrument over so that the same edge is now on the left of the instrument, and see if it just fits along the line. If it does, the straight-edge is accurate; if not, not.

'A straight line can have one position only when two points on it are fixed.'

The Angle.

The corner of an ordinary sheet of paper is formed by *two* straight lines which meet at the corner.

Put your straight-edge along one edge of the paper; the other edge of the paper crosses the straight-edge, and is part of a different straight line from the first edge.

The **figure** of two straight lines which end at a common point is an **angle** (the Latin word for corner).

Look at the figure of a set-square. How many corners has it? How many angles? Its figure is called a **triangle** (i.e. three-corner). The straight edges are its sides. How many sides has a triangle?

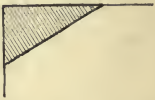
Look at a corner of your compass-box. How many angles are there at the corner?

Size of an Angle.

Divide a piece of paper into two parts. Cut or fold one piece across through a corner. Is your new angle at the corner greater or smaller than the old?

Since an angle can be greater or smaller, it is a magnitude—i.e. it has **size.**

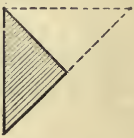
Try the angles of your set-square against a corner of your paper. Two of them should be smaller than the angle of the paper; the third should fit exactly. This angle is therefore **equal** to the angle of the paper.



Measure each of these equal angles with your protractor. How many degrees in each?

Bisector.

Fold across a corner of your paper so that the fold passes through the corner, and one side comes on the other. Mark the fold, and unfold and flatten out the paper.



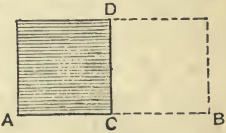
The fold divides the original angle into two angles; what do you know about these? Why? Measure each of them, and also the corner angle, with your protractor.

'A line which divides a figure into two equal parts is a bisector of the figure.'

The fold above is the **bisector** of the angle of the corner.

The Right Angle—Perpendicular.

Fold over an edge AB of paper so that one part CB of the edge comes exactly on to the other CA, and mark the fold CD. Unfold again and flatten out.



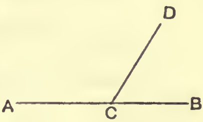
The fold forms two angles, one with each part of AB; and these are equal; so that the fold bisects the angle formed at C by the opposite parts CA, CB of the line. We therefore say that opposite parts of one straight line form an angle. This angle is generally called **two right angles,** and sometimes a **straight** angle.

The line CD which bisects the **straight** angle, or **two right angles,** at C is called a **perpendicular** to AB; and the angles formed by a straight line and a perpendicular to it are **right angles.**

Adjacent Angles—Opposite Angles.

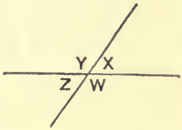
Lay the protractor with its edge along **AB** (last figure) and its centre at **C**. Measure the right angles at **C**. How many degrees in a right angle? in two right angles? in a straight angle?

Draw a straight line **AB**, 4″ or 5″ long. At a point **C** near its middle, draw another line **CD**, 3″ long.



Lay the base of the protractor along **AB**, centre at **C**, and measure the two angles formed. What is their sum? How many right angles?

Draw two straight lines as before, but make them cross. With protractor measure the angles marked **X, Y ; Y, Z ; Z, W**.



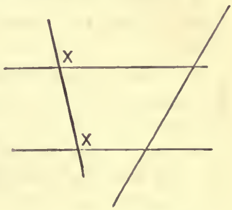
What is the **sum** of two **adjacent** (side by side) angles, as **X, Y**?

How does **X** compare with the **opposite** angle **Z**? **Y** with **W**?

'If two straight lines meet, two adjacent angles always make up two right angles; two opposite angles are always equal.'

Parallel Straight Lines—Direction.

Draw by set-square and straight-edge two parallel lines 4″ or 5″ long, and another line to cut these near their middle.



Measure the angles marked **X**; these are **towards the same parts,** because they face the same way.

Pick out and mark with **Y, Z,** &c., other pairs of angles **towards the same parts**; measure each pair.

Draw a second line across the parallels, and repeat.

'When a straight line crosses two parallels, the angles towards the same parts are always equal.'

Are any angles equal which are not towards the same parts?

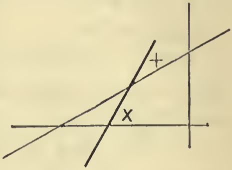
Two lines which make equal angles towards the same parts with any third line have the **same direction**; so that **'parallels are lines in the same direction.'**

It is shown in Chapter I. that parallel lines do not meet.

NON-PARALLELS.

Draw two straight lines 4″ or 5″ long to form an angle, and a third line to cut these.

Measure the angles marked +, **X**, towards the same parts.



Are these equal? Are other angles towards the same parts equal? Draw a second line crossing the two straight lines, and repeat.

Two lines which make unequal angles towards the same parts with any third line have **different** directions, and are **non**-parallel.

It is shown in Chapter II. that **non-parallel** lines in one plane meet.

EXAMPLES—II.

1. Mark two points 4″ apart on a sheet of paper; mark by folding the straight line joining them, and bisect this by folding again. Measure the two parts.

2. Draw a straight line **AB**, 2″ long. Mark its mid point **M**, and draw by set-square a perpendicular to **AB** at **M**. Take a point **C** on the perpendicular 1″ from **M**, and measure **CA**, **CB**. What do you notice? Take a point **D**, $1\frac{1}{2}$″ from **M**, and repeat.

3. Draw a straight line **AB**, $1\frac{1}{2}$″ long; mark points on it **C**, **D**, $\frac{1}{2}$″ from **A** and **B**. Draw perpendiculars at **C**, **D**. Are these parallel? Why?

4. Make a straight line **AB**; mark a point **P** on it, and draw a straight line through **P** at an angle of 73° to **AB**, and crossing it. Mark in degrees the four angles at **P**.

5. Draw two parallels **AB**, **CD** by set-square; mark a point **P** on **AB**, and draw **PQR** to make an angle of 45° with **AB**, and cross the parallels at **P**, **Q**. Write down in each angle at **Q** its number of degrees.

6. Fold over a corner of your paper so as to bisect the angle, and unfold again; mark a point **P** on the fold 2″ from the corner. Draw through **P** a perpendicular to the fold to meet the sides in **A**, **B**. Measure **PA**, **PB**.

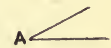
7. Draw two lines crossing at an angle of 51°. Write down the number of degrees in each of the other three angles.

8. Are two perpendiculars to a line always parallel? Why? Draw two perpendiculars **PC**, **QD** to a line **AB** from points **P**, **Q** on it; and draw **CE** perpendicular to **PC**. Is **CE** parallel to **AB**? Why?

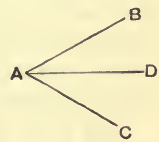
NAMES OF AN ANGLE.

An angle is conveniently named by a letter at its point, as the angle **A**; but sometimes three or more lines meet at a point, as, for instance, when an angle **A** is bisected by **AD**.

In that case any one of three angles at **A** *might* be meant by **A**; so we put a letter on each of the **sides** as well as at the **point** of the angles to distinguish them. The angles are then named



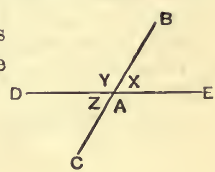
BAC, i.e. angle at **A** with sides **AB**, **AC**;
BAD, " **A** " **AB**, **AD**;
CAD, " **A** " **AC**, **AD**.



Note that the **point** of the angle is always indicated by the **middle** letter.

It is often convenient, however, to put a mark, such as **X**, **Y**, &c., in an angle to distinguish it. We then speak of the angle **X**, the angle **Y**, &c.

Using **A** as the point at which the lines **BC** and **DE** meet in the figure, write in three letters the angles **X**, **Y**, **Z**.



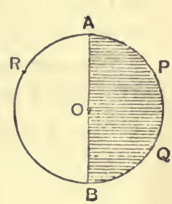
What is the value of **BAE** + **BAD**?
What angle is equal to **BAE**? to **BAD**?

THE CIRCLE.

Draw a circle, radius 1″. Mark the centre **O**. Mark points **P**, **Q**, **R** on the circle, and measure **PO**, **QO**, **RO**.

'All points on a circle are equidistant (i.e. equally distant) from the centre.'

Fold over a paper circle* so that the two parts exactly fit, and mark the fold **AB**. Measure the length of the fold, mark its mid point **O**, and unfold again.



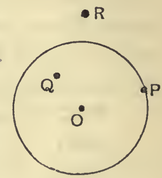
Mark points **P**, **Q**, **R** on the circumference, and measure the distances **PO**, **QO**, **RO**. Are they equal? What point is **O** of the circle? What fraction of **AB** is each of **AO**, **PO**, **RO**?

If you know that a point is 1″ from a given point **O**, on what curve must the point be? Mark a point **O**, and draw a curve every point of which is $\frac{3}{4}$″ from **O**.

* Two sizes of circular filter-papers or jam-covers will be found useful.

Locus.

A curve or line containing all points of a given kind, as, for example, all points 2″ from a given point, is the **locus** of those points. Sometimes a part only of a curve is the locus, because only *some* points, not *all*, on the curve are of the given kind.

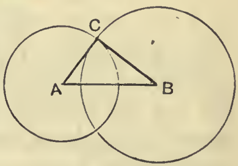


The locus of all points P which are $\frac{3}{4}''$ from O is the circle, centre O, radius $\frac{3}{4}''$.

Any point Q inside the circle is nearer to the centre, and any point R outside the circle is farther from the centre, than P.

This locus enables us to draw triangles whose sides are known.

Ex. Construct a triangle, sides 2 cm., 3 cm., 4 cm.



Set off one side AB, 4 cm.; then if C is the third point, C is 2 cm. from A, and is on the locus circle, centre A, radius 2 cm.; and it is also on the locus circle, centre B, radius 3 cm.

Draw these circles meeting at C. Join AC, CB.

ABC is the triangle.

EXAMPLES—III.

1. Draw a plan of the space on which a goat can feed when tethered by a rope 10 ft. long. (Use 1″ for 10 ft.)

2. Two posts are 10 ft. and 15 ft. from one end of a lane, and are 20 ft. apart. Represent on a plan (1″ to 10 ft.). If the lane is parallel to the line of posts, draw its plan. (Draw the lane as a straight line.)

3. Draw a triangle, sides $\frac{3}{4}''$, 1″, $1\frac{1}{4}''$, and measure the greatest angle.

4. Draw an equilateral (equal-sided) triangle, each side 2·5 cm. Measure an angle.

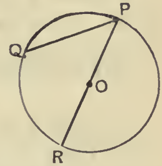
5. Draw an isosceles triangle (two sides equal), the equal sides 3·2 cm., the third side 1·8 cm.

6. Draw a circle, radius 1″, and mark a point P on it. Find two other points Q, R on it $\frac{3}{4}''$ and $1\frac{1}{2}''$ from P. Can you find a point on the circle 3″ from P?

7. Draw a straight line AB, and a point P about 1″ from AB over its middle point. Find a point in the straight line $1\frac{1}{2}''$ from P. How many such points can you find?

The Chord of a Circle.

Draw a circle, centre **O**, radius 2 cm. Mark two points **P**, **Q** on it, and join **PQ**.



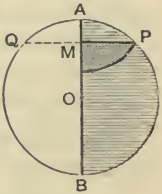
The straight line **PQ** is a **chord** of the circle; either part of the curve from **P** to **Q** is an **arc** of the circle.

Draw a chord **PR** through the centre **O**.

A chord of a circle through the **centre**, as **PR**, is a **diameter.**

If a circle is folded over a diameter, one side fits exactly on the other, so that **a diameter bisects the circle.**

Fold over a paper circle so that one side fits exactly on the other, and mark the fold **AB**.



AB is a diameter, and its mid point **O** the centre.

Fold over again at a point **M** in **AB** so that the part **MA** of the first fold fits exactly along **MB**, and mark the new (double) fold **MP**.

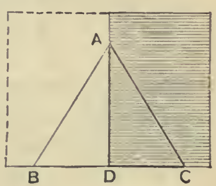
Unfold and flatten out as in the figure.

What are the angles at **M**? Is **MP** = **MQ**?

'A chord perpendicular to a diameter is bisected by it.'

The Isosceles Triangle.

Fold over a piece of paper at a point **D** in a straight-cut edge, so that one part of the edge **DB** comes on the other **DC**.



Cut across **AC** so as to form a twofold triangle, and unfold; ∴ side **AB** = ?
The figure formed is an **isosceles*** triangle **ABC**.

The fold **AD** is the bisector of angle **A**; what are the angles at **D**?

Also angle **B** exactly fits angle **C**;
∴ angle **B** = ?

'The angles at the base of an isosceles triangle are equal.'

If **BD** is 2″, calculate **DC**. If the angle **BAC** is 37°, calculate angles **BAD**, **DAC**.

If angle **B** is $71\frac{1}{2}°$, what is **C**?

If a triangle **ABC** is equilateral, what angles are equal?

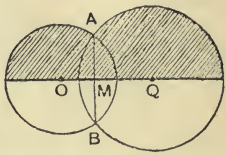
Ex. Make an isosceles triangle **ABC**, equal sides **AB**, **AC** $1\frac{1}{2}''$, third side 2″. Measure its angles. Which are equal?

* That is, equal-sided.

Two Circles.

Take two paper circles, slide one over the other; in how many points do the two curves meet?

'Two circles never meet in more than two points.'



Fold each so as to mark a diameter, place them to overlap with these diameters in one line, and fasten them together with gummed paper.

Fold both over simultaneously about the common diameter OQ; one point B where they meet comes exactly on the other point A.

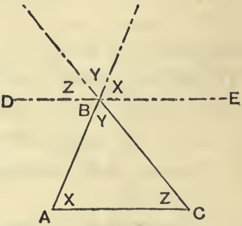
Unfold; the chord AB is bisected at M, and the angles at M are right angles.

AB is the **common** chord of the two circles.

'The line of centres of two circles bisects at right angles their common chord.'

The Angles of a Triangle.

Draw a triangle ABC; and at B make DBE parallel to AC, and prolong CB, AB to cross DE, making the angles X, Y, Z.



Then angle X = A (same parts),
Z = C "
Y = B (opposite angle).

Set the protractor with its base along DE, and its centre at B, to cover the three angles X, Y, Z.

X + Y + Z = 180°.

∴ A + B + C = ?

'The three angles of a triangle always make up two right angles.'

If ABC is equilateral the three angles are equal, and together make up 180°.

But 3 × 60 = 180; and therefore

'Each angle of an equilateral triangle is 60°.'

EXAMPLES—IV.

1. Construct an isosceles triangle, base AB 4 cm., sides AC, BC each 3 cm.

2. Construct a point C, 3 cm. from A and from B, A and B being 4 cm. apart. Is the construction the same as that of Ex. 1? Why? How many such points do you find?

3. Construct an isosceles triangle, equal sides $1\frac{1}{2}''$, their angle 40°. Calculate the other angles.

4. Draw an equilateral triangle, side $1\frac{1}{4}''$. Measure one of its angles. What ought it to be? What angle can you thus construct?

5. Two angles of a triangle are 33° and 106°. What is the third angle?

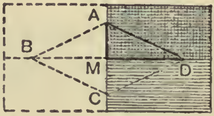
6. Take two points A, B, 5 cm. apart. Construct two points C, D, each 3 cm. from A and 4 cm. from B. Measure CD and mark its mid point.

7. Find two points C and D each 3 cm. from A and B, when A and B are 4 cm. apart. What distance is A from C and D? Mark the mid point of AB.

8. Can the three angles of a triangle be 72°, 94°, 30°? Why? If the first two are correct, what ought the third to be?

The Rhombus.

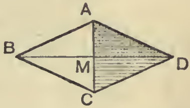
Fold over a piece of paper, and mark the fold AC. Fold over again so that one part of the fold MC comes on the other MA, and mark the second fold MD.



Cut across AD so as to cut out a fourfold triangle AMD; unfold and flatten out into the four-sided figure ABCD.

What do you know about the sides of this figure?

An equal-sided four-sided figure is a **rhombus.** AC, BD are **diagonals.**



What do you know about the angles ADB, CDB? About the angles at M? About the parts MA, MC of diagonal AC?

'The diagonals of a rhombus bisect its angles, and bisect each other at right angles.'

RHOMBUS AND BISECTORS.

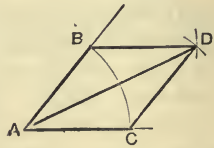
It is easy to construct a rhombus when its side and one angle, or its side and a diagonal, are given.

Ex. (i.). 'Construct a rhombus, angle 46°, side $\frac{3}{4}''$.'

Make angle **A**=46°; with centre **A**, radius $\frac{3}{4}''$, draw arc of circle **BC** to cut the sides of angle **A** in **B, C**.

With centres **B, C**, same radius ($\frac{3}{4}''$), draw arcs cutting in **D**; join **BD, CD**.

ABDC is the rhombus.



Also, since the diagonal **AD** bisects the angle **A** of the rhombus, a similar construction enables us to bisect a given angle **A**. Thus:

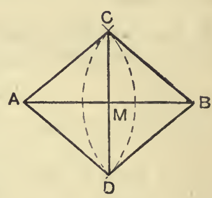
Draw a circle, centre **A**, *any* radius **AB**, cutting the sides **AB, AC** of the angle in **B, C**; with centres **B, C**, *same* radius, draw arcs **D**.

AD is the bisector of angle **A**.

Ex. (ii.). 'Construct a rhombus, one diagonal 1″, side $\frac{3}{4}''$.'

Make **AB** 1″; with centres **A, B**, radius $\frac{3}{4}''$, draw arcs of circles cutting in **C, D**.

ACBD is the rhombus.



Also, since the diagonal **CD** bisects **AB** at right angles, a similar construction enables us to draw a line **CD** bisecting at right angles (at **M**) a given line **AB**. Thus:

Draw circles of equal radii, centres **A, B**, to cut in **C, D**; the line **CD** bisects **AB** at right angles in **M**.

THE RIGHT BISECTOR.

The line bisecting another line **AB** at right angles is the **right bisector** (i.e. the **perpendicular** bisector) of **AB**.

If **C** (last figure) is any point equally distant from two points **A, B**, the triangle **CAB** is isosceles, and folds over so that the bisector **CM** of angle **C** is also the right bisector of **AB**. Thus:

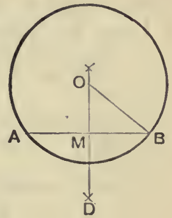
'The locus of points equidistant from two fixed points A, B is the right bisector of the line AB.'

By the aid of this locus we are able to draw circles to pass accurately through two given points.

Ex. What kind of triangles are **ABD, ACD** in the upper figure? **ABC, ABD, ACD, BCD** in the lower figure?

Circle through Given Points.

Ex. (i.). 'Draw a circle, radius $\frac{1}{2}''$, to pass through two points A, B, $\frac{3}{4}''$ apart.



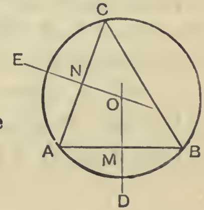
Draw and produce **DM**, the right bisector of **AB** (as on last page).

With centre **B**, radius $\frac{1}{2}''$, draw arc of circle to cut **DM** in **O**.

Draw circle, centre **O**, radius $\frac{1}{2}''$, through **A, B**.

Can you do this more simply?

Ex. (ii.). 'Draw a circle through the three angular points A, B, C of a triangle.'



Draw and produce **DM**, the right bisector of **AB**;
" " **EN**, " " **AC**,
to meet **DM** in **O**.

With centre **O**, radius **OA**, draw the circle through **A, B, C**.

This circle is the **circumcircle** of triangle **ABC**.

EXAMPLES—V.

1. Draw a rhombus, ang. 56°, side 3 cm. Measure its diagonals.

2. Draw angles of 72°, 84°, 108°, 156°, and bisect them. Measure one of the parts in each case.

3. Draw a rhombus, angle 66°, side $1\frac{1}{4}''$. Test with your set-square to see if opposite sides are parallel. Are they? Calculate the angles made by the longer diagonal and the sides.

4. Draw a rhombus, diagonal 4·2 cm., side 3·5 cm. Calculate the sum of its four angles. (Use the two triangles.)

5. Draw lines $1\frac{1}{2}''$, $2\frac{1}{2}''$, $2\frac{3}{4}''$ long, and draw their right bisectors. Measure one part in each case.

6. Draw a rhombus **ABCD**, angle **A** 60°, side 1″. Draw and measure perpendiculars from **A, B** on the side **CD** (produced sufficiently).

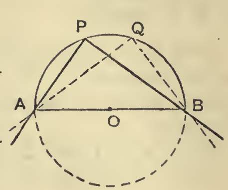
7. Make a straight line **AB**, 6 cm. Draw a circle, radius 5 cm., to pass through **A, B**. How far is its centre from the line **AB**?

8. Draw a triangle, $a=1''$, $b=1\frac{1}{4}''$, $c=1\frac{1}{2}''$; and draw its circumcircle.

9. Make an angle **BAC**, 30°. Make **AC**=2 cm., **AD**=5 cm., along **AC**; and draw a circle to pass through **C, D** and have its centre in **AB**. (Draw the right bisector of **CD** to meet **AB** in the centre of the circle.)

Angle in a Semicircle—Angles in an Arc.

Draw a semicircle **APB**, centre **O**, radius 1″, diameter **AB**. Take a point **P** on the circle.



Draw **PA**, **PB** to 3″ from **P**.

Measure the angle **APB**. How many degrees?

Take a second point **Q** on the circle and repeat.

The angle **AQB** is ? degrees.

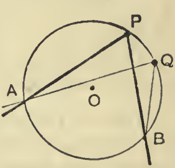
Fix two pins (or divider points) at **A**, **B**, place the right angle of your set-square on the semicircle so that its sides are close against the pins at **A**, **B**; the point of the right angle is on the circle.

Repeat in several other positions; the point of the right angle is always on the circle.

'The angle in a semicircle is a right angle.'

Draw a circle, radius 1″, centre **O**. Place in it a chord **AB**, $1\frac{3}{4}''$; take points **P**, **Q** as before, and measure the angles **APB**, **AQB**.

APB is ? **AQB** is ?



Put the 60° angle of your set-square on the circle **APB**, with its sides against pins at **A**, **B**.

The point of the angle comes on the circle.

Repeat in several positions. The point of the angle is always on the circle.

Repeat the above process, using the 30° angle of your set-square, a circle of 1″ radius, and a chord of 1″.

'Angles in the same arc of a circle are equal.'

This gives us another locus—viz. that of the point of a given angle whose sides traverse fixed points. This locus is only *part* of a circle (or more strictly *two* parts of circles, one on each side of **AB**).

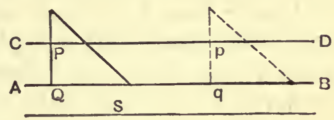
In each case above measure the angle formed at the centre by radii to the pin-points. Compare with that in the corresponding arc.

'An angle at the centre of a circle is double the angle at the circumference on the same arc.'

POINTS EQUIDISTANT FROM A LINE.

There is still another locus which is very useful.

Draw a straight line AB, and a parallel CD about $1\frac{1}{2}''$ from it.



Bring a straight-edge S along AB, so that the instrument lies on the side of AB away from CD. Put one side of the right angle of the set-square on AB, and mark on the set-square with a pencil the point P where the other side crosses CD.

Slide the set-square along the straight-edge (which must be held firm); P always comes on the parallel CD, so that the distance of any point on CD from AB is always the distance PQ from P to the corner Q of the set-square.

'The locus of all points equidistant from a fixed line is two parallels to the line at that distance.'

Where is the second parallel?

We can easily construct these parallels.

Ex. 'Construct a parallel to AB at a distance of 3 cm.'

Make AC perpendicular to AB, 3 cm. long.
Through C draw CD parallel to AB.

EXAMPLES—VI.

1. Draw 4 circles, radius 4 cm. Place chords 7 cm., 4 cm., 3 cm., 2 cm. in the successive circles; and measure an angle in the larger arc in each case.

2. Take AB, 4 cm. Find a point P such that angle APB is a right angle, and distance AP = 2 cm. (Use semicircle.)

3. Construct a line AB, and a point C $1\frac{1}{4}''$ from it. Find a point P $1\frac{1}{2}''$ from C and $1''$ from the line BA.

4. One side of a set-square slides along a straight-edge. What is the locus of the opposite point of the set-square?

5. Construct an angle BAC of 72°. Construct a point P on the side AB, $1\frac{1}{4}''$ from AC.

6. Construct a semicircle, centre O, diameter AB, radius 2 cm. Draw OC perpendicular to AB, meeting the circle in C. Join AC, BC. What angle is ACB? Compare the sides AC, BC.

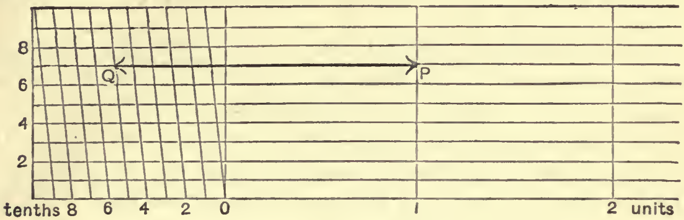
The experiments which show that all angles in certain arcs are equal are excellent examples of the way in which new facts may be discovered by experiment; but as we cannot try the experiment in all possible cases, it is natural to see if we can find some way to convince ourselves that the result must be true in all cases. This is done by inventing a *theory* to account for all the facts; the accuracy of our invented theory is then put to the proof from time to time by testing experimentally the truth of new results which it establishes. If any of these are clearly not true, our theory must be erroneous or incomplete.

Statements based on a theory are called **theorems**, and can be proved by the original definitions, or by other theorems previously established. Such proof is called **formal**; and the succeeding chapters of this book constitute the more elementary part of **formal** as distinct from **experimental** geometry. Practical methods of drawing and measuring are derived from theorems and embodied in **constructions.** The proofs of Theorems 2, 3, and 6 in Chapter I. are, however, experimental, the theoretical proofs being given at the end of Chapter IX.

The fundamental definitions of formal geometry are those of the plane, straight line, angle, and direction or parallels; in addition to these we require notions of length, of extent of surface and volume—i.e. of 'space'—of magnitude, of position, of movement or change of position, and of shape or form. We also *assume*—i.e. take it for granted—that a body or figure may be moved without any change (of shape, &c.) except that of position, and that any magnitude (length, area, &c.) may be so multiplied as to exceed any other like magnitude (e.g. a mile may be so multiplied as to exceed the distance from the earth to the sun); and we also assume the general notions of equality and inequality.

The theory of geometry is very old, dating back at least 2500 years to Thales; it has been developed (to give a few names) among the ancients by Pythagoras, Euclid, Archimedes, Ptolemy, &c., and among the moderns by Pascal, Newton, Poncelet, Monge, Carnot, Cremona, &c.

We conclude this section with a brief account of scales,* because of their connection with our instruments of measurement.



The **diagonal scale** enables small fractions of an inch or half-inch or centimetre to be measured more accurately than the plain scale allows.

The base line is divided into a plain scale of units and, say, tenths; and perpendiculars are drawn through the unit divisions 0, 1, 2, &c. Ten parallels to the base line are drawn at equal distances; and the tenth divisions of the base line are joined diagonally to those of the top line.

This enables us to measure tenths of tenths—i.e. hundredths of the unit.

'Measure a length PQ by an inch diagonal scale, showing hundredths.'

Adjust the divider points to PQ, bring one of them along a unit perpendicular and the other in the divided unit on a parallel through the first.

Move the divider points, always parallel to the base line, until the Q end is on a diagonal as well as a parallel, keeping P always on a unit perpendicular.

In the example P is on the 1st unit, $PQ = 1''$......

" Q " 5th diagonal, $PQ = 1{\cdot}5''$......

" Q " 7th parallel, $PQ = 1{\cdot}57''$.

Similarly we can set the divider to a length of $1{\cdot}57''$.

Note. If the base unit is divided into 12 parts, and there are 8 parallels, the diagonal scale shows eighths of twelfths—i.e. ninety-sixth parts—and so on.

* The rest of this section may be postponed by beginners. The use of the diagonal scale is required for some of the examples in Chapter III.

Scales are constructed by setting off the unit a suitable number of times along a straight line, and dividing one unit by construction (p. 4) into the required number of parts, as in the figure on p. 1.

In constructing a diagonal scale, a drawing-board should be used; a T-square for the base line and its parallels, and a set-square on the T-square for the perpendiculars. When the unit on the base line has been divided, the corresponding points on the top line can be marked from these by moving the set-square along the T-square; the top and bottom divisions must then be joined *diagonally*.

Drawing 'to Scale' and Comparative Scales.

In maps and plans, objects are rarely represented in their actual size; for drawing such plans a scale is made or used in which the actual unit (1 mile, 1 yd., &c.) is replaced by a more suitable length (1 in., $\frac{3}{4}$ in., &c.); and the plan drawn by means of this scale is said to be 'to a certain scale'—e.g. 1″ to the mile, $\frac{3}{4}''$ to the yard, a scale of 1 : 20, &c.

The fraction $\frac{\text{scale unit}}{\text{actual unit}}$ is called the **representative fraction** of the scale. Thus, if $\frac{3}{4}''$ represent 1 yd.,

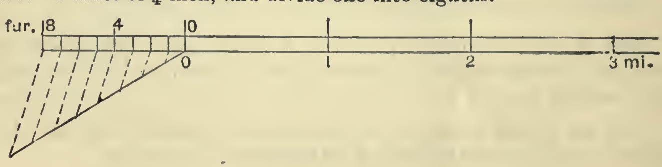
$$\text{R.F.} = \frac{\frac{3}{4}}{36} = \frac{1}{48};$$

and for 1″ to the mile, $\text{R.F.} = \frac{1}{1760 \times 36} = \frac{1}{63360}$.

Ex. Draw a scale of miles up to 4 miles to show furlongs, $\text{R.F.} = \frac{1}{84480}$.

$$\begin{aligned} 1 \text{ mile is represented by } & \frac{1}{84480} \text{ mile} \\ &= \frac{1760 \times 36}{84480} \text{ inch} \\ &= \tfrac{3}{4}''. \end{aligned}$$

Set off units of $\frac{3}{4}$ inch, and divide one into eighths.



In measuring distances on foreign maps, it is sometimes necessary to convert measures from one scale to another—e.g. from kilometres to miles. In this case a **comparative** scale of kilometres can be constructed from the scale of miles or *vice versâ.*

Ex. Given a scale of $1\frac{1}{2}''$ to the mile, construct a comparative scale of kilometres, showing tenths, up to six kilometres.

1 kilometre $=39370''$, 1 mile $=63360''$.

$\therefore$ 1 kilometre is represented by $\frac{3937}{6336} \times 1\frac{1}{2}'' = \cdot 93''$.

Set off units of $\cdot 93''$ and divide one into tenths.

EXAMPLES—VII.

Scales.

1. Make a scale of miles $\frac{3}{4}''$ to the mile up to 6 miles, showing furlongs. Indicate on it a length of 3 miles 7 fur.

2. Using scale 1, make a plan of a trapezium-shaped park, parallel sides $2\frac{1}{4}$, $3\frac{1}{2}$ miles, their distance apart $1\frac{3}{4}$ miles, another side $2\frac{1}{2}$ miles.

3. A French map is drawn 2 cm. to the kilometre. Make a comparative scale of miles. (1 mile $=1\cdot 609$ km. ; 1 km. $=1000$ metres $=100{,}000$ cm.) Write down R.F.

4. Make a scale of yards, $\frac{3}{4}''$ to the yard, up to 1 pole, showing feet. Divide diagonally to show inches. Give R.F.

5. Make a scale of poles up to 1 chain, showing yards, R.F. $\frac{1}{198}$.

6. From a scale of miles $\frac{5}{8}''$ to the mile, up to 8 miles, construct a scale of geographical miles (60 geographical miles $= 69\cdot 1$ miles), showing tenths.

7. On a map a distance of 3 miles 2 fur. is represented by $1\frac{1}{12}''$. Find the R.F., and show a line representing 5 miles 5 fur.

8. Make a scale of Russian versts from a scale of miles $1\cdot 2''$ to the mile. (1 verst $=\cdot 663$ mile.) Calculate R.F.

9. Make a scale of furlongs up to 1 mile, R.F. $\frac{1}{1584}$. Divide diagonally to show poles.

10. A yachting course is 9 km. E., 12 N., and 15 home. Calculate the distance in miles, and represent on a map $\frac{1}{2}''$ to the mile. (1 km. $=\frac{5}{8}$ mile.)

11. Make a scale of kilometres up to 5 km., 3 cm. to the km. ; calculate the R.F., and represent a distance of $2\cdot 57$ km.

12. Make a scale 25 cm. to the kilometre, up to $\frac{1}{2}$ km., showing tenths, and show metres by diagonal division. (1 km. $=1000$ metres.)

ABBREVIATIONS.

These may be referred to as they occur in the text.
The following symbols are used:

$\therefore$ therefore; $\because$ because;
$=$ is or are equal to, $\neq$ is or are not equal to;
$>$ " greater than, $\not>$ " greater than;
$<$ " less than, $\not<$ " less than;
$\gtreqless$ is greater than, equal to, or less than;
$\equiv$ is congruent to; $|||$ is similar to;
$\parallel$ is parallel to; $\perp$ is perpendicular to;
$\triangle$ area of triangle; $\square$ area of parallelogram.

Obvious abbreviations of words are used:

str., straight; perp., perpendicular; parl., parallel; eql., equal;
hyp., hypotenuse; alt., altitude, alternate; chd., chord;
tangt., tangent; bisr., bisector;
ang., angle; rt. ang., right angle;
compt., complement; suppt., supplement;
fig., figure; poln., polygon; sq., square;
tr., triangle; quadl., quadrilateral;
parm., parallelogram; rect., rectangle;
circ., circle; semcle., semicircle.
opp., opposite; adj., adjacent;
propl., proportional; simr., similar.
simly., similarly; simde., similitude;
Def., definition; Th., theorem; Cor., corollary;
Constr., construction; Ex., example, &c.;
line is used for straight line;
a, b, c, sides of triangle ABC;
A, B, C, angles of triangle opp. a, b, c respectively;
s = semi-sum of sides $= \dfrac{a+b+c}{2}$;
R, radius of circumcircle;
r, " incircle;
r_a, r_b, r_c, radii of ecircles.

PLANE GEOMETRY.

CHAPTER I.

PRELIMINARY DEFINITIONS AND THEOREMS.

Geometry treats of the position, form, and size of bodies by means of figures.

The fundamental assumption or postulate of geometry is that figures or bodies may be supposed to be moved without changing either form or size.

Definition 1.—A **surface** is the boundary of a body or part of a body.

Ex. Top of a table, outside of a book.

Definition 2.—A **line** is the boundary of any part of a surface.

Ex. Edge of a table, of a page of a book.

Definition 3.—A **point** is the boundary of any part of a line.

Ex. Corner of a table, of a page of a book.

Points may be marked by a cross (×) or a dot (.).

Note. If a point moves, its path is a line, and it is said to generate the line—e.g. a pen-point traces or generates a line on paper. A line of this kind is often called a locus.

Similarly, a line moving generates a surface, as a hand of a clock (considered as a line); a knife-edge cutting an apple generates the cut surface.

Surfaces are recognised as of many kinds, rough, smooth, hard, soft, &c.; in geometry we are concerned with their shape and size only, as round, flat, conical, &c.

Ex. Name bodies of round, flat, conical surfaces.

The most important surface is an ideally flat or level surface. Many surfaces are nearly so—that of still water, of a well-finished table, slate, or drawing-board; the best examples are ordinary plate-glass, certain machine-made steel 'planes,' and 'optically plane' glass.

THE PLANE.*

Take a post-card, mark two points on one side, lay a straight-edge against them, and with the point of a knife along the straight-edge score the card along a straight line, taking the line right across the card. Be careful not to cut the straight-edge, nor to cut right through the card.

Now fold over the card so that one part fits closely on the other. A surface which can be folded over in this manner about *any two* points in it is a plane surface.

Take two post-cards, and place one on the other; notice that you can slide or turn one on the other, and that they coincide—i.e. exactly fit—within their common boundary. This is always true of two plane surfaces, or two parts of a plane surface.

Lay a post-card on a slate, a drawing-board, or a larger sheet of cardboard; its surface will coincide everywhere with that of the other as it slides over it; hence there is a plane surface—e.g. that of the board—greater than that of the post-card. Similarly we could find a sheet of plate-glass with a plane surface greater than that of the board, and so on; and there is every reason to believe that however great a plane surface is, a greater one is possible: we say, then, that a plane surface can be produced or extended indefinitely—or, briefly, that it is infinite.†

Note. It is only the **surface** of the card that coincides with the **surface** of the board or sheet of glass; mentally, we picture the **surface** as having position and as having sides—e.g. upper and under; but being a boundary only, it has no substance.

* A few post-cards and some tracing-paper will be useful in addition to ordinary paper. Sheets of plate-glass would, of course, be better than post-cards for showing coincidence of two planes.

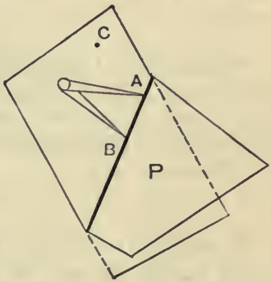
† I think it possible that this fact is included in the folding definition of plane, but I have not so far succeeded in proving this.

We have thus arrived at the following properties of a 'plane' surface: it folds over about any two points, it is infinite, and it can be placed so as to coincide with any other plane.*

The fold of a plane when folded over is the most important kind of line—viz. a straight line. A well-made steel straight-edge gives the best example of this.

The Straight Line.

Carefully fold a smooth sheet of paper,† and make the fold as precise as possible. Note that the fold extends right across the paper.



Unfold and lay the sheet on a flat surface (board or slate), fix two points A, B of the fold with divider points, and hold down a third point C, not on the fold, with a finger.

Now turn the free part P of the paper in any way. Notice that the fold does not move, but divides the free part of the surface from the fixed part. Now keep only the points A, B fixed and turn either or both parts of the paper without pulling; note that no part of the fold moves.

Fold a second sheet and fix two points A, B on both folds simultaneously; the two folds coincide, and neither moves as we move the parts of either or both sheets. This is true of all straight lines—viz. that when two points of one are fixed, the straight line has one position only. Also, as the fold extends to the boundary of the plane, by extending the plane we extend the straight line; hence a straight line is also infinite.

Now place your two folds with only one point, D say, common; note that they cross. Two straight lines (produced sufficiently) with one point common always cross.

We now present these properties of plane and straight line in the form of definitions and theorems.

* We shall see presently that the last fact is a consequence of the others and of the properties of the fold.

† The paper should be fairly stiff.

Definition 4.—A **plane** is a surface, infinite in extent, which can be folded about any two points of the surface so that one part lies entirely on the other.

Definition 5.—A **straight line** is the fold of a plane about any two points in it, and is infinite in extent.

The limited straight line from a point A to a point B is the **join** of A, B.

Theorem 1.—'A straight line can be drawn through any two points in a plane, and lies entirely in the plane.'

Theorem 2.—'Any two straight lines coincide entirely when two points of each coincide.' *

These two theorems follow at once from the properties of a fold shown above.

Theorem 3.—'Every finite straight line has a mid point about which it can be reversed so that the ends are interchanged.' *

Mark two points A, B on a fold of paper.†

Fold the paper over until one of them A comes on the other B, and mark the new fold, crossing the old fold at M.

Thus the part MA of the line folds over on to MB. Similarly MB will fold on to MA; hence MA = MB.

Thus the line can be reversed on itself so that A and B are interchanged, the mid point M remaining unmoved.

It follows also that the straight line can be turned in the plane about M so as to be reversed.

Note. The second fold in this case is perpendicular to the first (see Def. 9 and 11), and makes right angles with it.

* Formal proof is given in a note after Chapter IX.

† Tracing-paper may be used for a straight line drawn in ink or pencil.

EXAMPLES—VIII.

1. Draw a straight line AB with the aid of a fold of paper. Test it by placing the folded sheet first on one side and then on the other side of the line.

2. Test in the same way all your straight edges (set-squares and scales).

3. Mark two points in a straight line AB ; measure their distance with divider and scale.

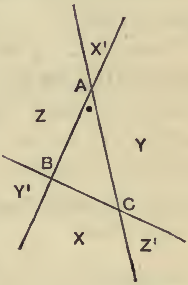
Interchange the divider points and measure again. Are both measures the same? Why?

4. Stretch a piece of thread along the paper. Test with a straight-edge. Is it straight?

5. Bisect the edge of a piece of paper by folding over. Test by measuring.

Properties of the Plane.

Lay any plane P (of tracing-paper) on another Q, and prick through three points A, B, C not in one straight line, so that these points are in both planes.



The complete straight lines AB, BC, CA lie in both planes. (Thh. 1, 2.)

If now a straight edge* turns in the top plane about a pin fixed at A, so as to sweep out the space XX′ in that plane, it will cross some point of CB at every stage of the process, and hence the complete line represented by the straight edge lies always in both planes.

Thus the whole space XX′ lies in both planes.

Similarly YY′ and ZZ′ lie entirely in both planes, so that the planes P, Q coincide altogether. Hence :

Theorem 4.—'Any two planes coincide when three points of each, not in one straight line, coincide.'

This theorem remains true when the two sides of either plane are interchanged. Hence :

Theorem 5.—'A plane may be reversed upon itself.'

* Supposed infinitely long.

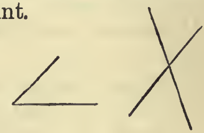
PLANE FIGURES.

Definition 6.—Any group of points, lines, surfaces, is a **figure.**

Definition 7.—A **plane figure** is a figure whose points and lines are all in one plane.

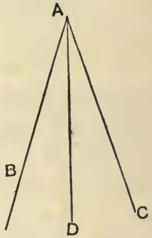
Definition 8.—An **angle** is a plane figure formed by two straight lines which terminate at a common point.

If two lines cross they form four angles at the point of crossing.



Theorem 6.—'Every angle has a bisector about which it can be reversed so that its sides are interchanged.' *

Draw an angle A on tracing-paper, fold over about A so that one side AB comes on the other AC, and mark the fold AD.



Thus the angle BAD folds on to CAD, about the bisector AD.

Similarly CAD folds on to BAD;
hence CAD = BAD.

Thus the angle BAC can be reversed on itself so that AB and AC are interchanged, the bisector AD remaining unmoved.

Note. If AB, AC are opposite parts of one straight line, there is still a bisector AD.

Definition 9.—A **perpendicular** to a straight line at any point in it is the bisector of the angle formed by opposite parts of the line at that point.

To obtain this, fold over a straight edge of paper about a point A in it, so that one part AC of the line folds on to the other part AB. The new fold is the perpendicular.

Definition 10.—The **right bisector** of a straight line is the perpendicular to it at its mid point.

Ex. Construct, by folding, the right bisector of an edge of a sheet of paper.

* Formal proof is given in a note after Chapter IX.

Theorem 7.—'One only perpendicular exists to a straight line at each point in the line.'

For there is one only bisector of the angle of opposite parts AB, AC of the line at each point A of it (next fig.).

Definition 11.—A **right angle** is an angle formed by a straight line and a perpendicular to it.

Theorem 8.—'All right angles are equal.'

If $AD \perp BC$, and $EH \perp FG$,
place the figure FEHG on BADC, so that FG comes on BC, and E on A.
∴ the bisector EH comes on AD;
∴ rt. ang. HEG coincides with rt. ang. DAC;
i.e. rt. ang. HEG = rt. ang. DAC.

Similarly any two right angles are equal.

Definition 12.—An **acute angle** is less than a right angle.

An **obtuse angle** is greater than one, less than two right angles.

Definition 13.—The **complement** of an angle is its defect from one right angle. (Compt. of $A = 90° - A$.)

The **supplement** of an angle is its defect from two right angles. (Suppt. of $A = 180° - A$.)

EXAMPLES—IX.

1. Take a piece of paper with straight-cut edges. Fold over a corner so that one edge comes on the other, and mark the fold. What is the fold called? Measure the angles thus formed.

2. Fold over a straight edge of paper so that one part of the edge comes on the other, and mark the fold. What is the fold called? The two angles thus formed? Measure them.

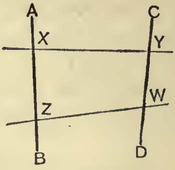
3. Draw angles of 20°, 54°, 90°, 135°, 167°. Which are acute, right, obtuse?

4. Write down the complements of 17°, 56°, 72°, and the supplements of 36°, 72°, 153°.

5. Draw the right bisector of a straight line AB, 2″ long. Compare the distances from A, B of several points in it. (Bisect the line at M by parallels as on p. 4, and draw a perp. as on p. 3, or use Ex. (ii.), p. 14.)

6. Through how many right angles does the minute-hand of a clock turn in a day (24 hrs.)?

Definition 14.—Two straight lines in a plane have the **same or different directions** according as they make with any third line whatever equal or unequal angles towards the same parts.



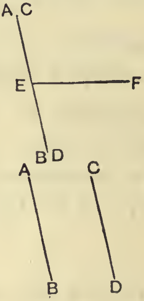
Thus AB and CD have the same direction if X = Y, and then also Z = W;*
and AB and CD have different directions if X ≠ Y, and then also Z ≠ W.†

Definition 15.—**Parallel straight lines** are coplanar lines having the same direction.

(**Coplanar** lines are lines in one plane.)

Ex. Two perpendiculars to a given straight line are parallel.

Theorem 9.—'Two parallels either coincide or do not meet.'



Either (i.) the parallels AB, CD have a common point E, as in fig. (i.);
then if EF is a third line in their plane,
ang. CEF = AEF, towards same parts;
∴ CE coincides with AE;
i.e. CD coincides with AB;
or (ii.), the parallels AB, CD have no common point;
i.e. they do not meet.

EXAMPLES—X.

1. If a set-square moves with one edge along a fixed straight line, two positions of a second edge are parallel.

2. Draw two parallels, and a line cutting one at 70°; measure the angles it makes with the second parallel. Which are equal?

3. Draw two straight lines meeting, and a third line to cut both; measure the angles it makes with them. Are any equal?

4. Draw two lines at an angle of 53°, and two parallels to these. Measure the angle of the latter towards the same parts.

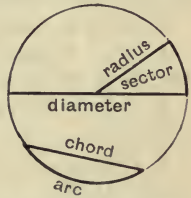
5. Make a four-sided figure of two sets of parallels; compare its angles, and its opposite sides.

6.† 'If *a*, *b* are parallels, and also *c*, *d* are parallels, show that the angle *a*, *c* = angle *b*, *d* towards the same parts.'

* This definition includes Euclid's parallel axiom. † Important.

The Circle.—If a straight line turns in a plane about a fixed point in the line, so as to sweep out the whole space of four right angles round the point, any point of the straight line describes a closed curve enclosing the fixed point.

Definition 16.—A **circle** is a closed plane curve of which all points are equally distant from a fixed point in its plane called the centre.



A **radius** is a straight line from centre to curve.

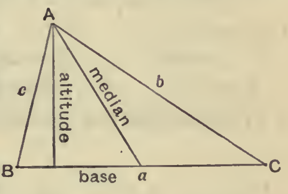
A **chord** is the join of two points on the curve.

A **diameter** is a chord through the centre; it is double any radius, and divides the circle into two **semicircles.**

An **arc** is a part of the curve.

A **sector** is a part of the circle bounded by two radii and the intercepted arc.

Definition 17.—A **triangle** is a closed plane figure formed by the joins of three points not in a straight line.



The three points A, B, C are **vertices,*** their joins AB, BC, CA the **sides,** and the three interior angles the **angles,** of the triangle.

The angles may be denoted by the capital letters A, B, C, and the sides opposite these by the italic letters *a*, *b*, *c*, when no confusion arises from doing so.

An **altitude** † of a triangle is the length of a perpendicular from a vertex to the opposite side, which is then called the **base.**

A **median** is a straight line from a vertex to the mid point of the opposite side.

Definition 18.—An **isosceles** triangle has two equal sides.

An **equilateral** triangle has three equal sides.

A **right triangle** has one angle a right angle, and the side opposite this is the **hypotenuse.**

An **obtuse** triangle has one angle obtuse.

An **acute** triangle has **three** acute angles.

* Singular, vertex ; plural, vertices or vertexes.

† Or perpendicular, when used of the direction only, without reference to magnitude.

EXAMPLES—XI.

1. Draw a circle, 1″ radius, and place in it a chord $1\frac{1}{2}''$ long. Cut off an arc whose chord is $1\frac{1}{4}''$ long. Measure the longest chord of the circle.

2. Draw an isosceles triangle, given $a=b=1\frac{1}{2}''$, $c=\frac{3}{4}''$. Measure the altitude from **C** to **AB**.

3. Draw an isosceles triangle, given $\mathsf{C}=34°$, $a=b=2\frac{1}{2}''$. Measure the altitude from **C** to **AB**.

4. Draw an equilateral triangle of $1\frac{3}{4}''$ side. Measure one angle.

5. Draw a right triangle, hypotenuse 2″, one side 1″. Measure the median of the hypotenuse, and the larger acute angle.

6. Draw a triangle, altitude 1″, base $1\frac{1}{4}''$, one side $1\frac{1}{2}''$. Measure the third side.

7. Draw a triangle, $a=2\frac{3}{4}''$, $\mathsf{B}=42°$, $\mathsf{C}=64°$. Measure the altitude from **A** to **BC**.

Definition 19.—A **quadrilateral** is a closed plane figure formed by the joins of four points,* two and two, and two and two.

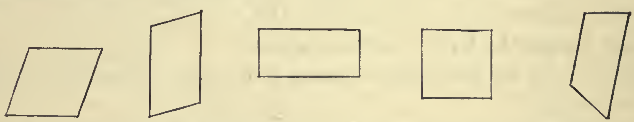
A **rhombus** is an equal-sided quadrilateral.

A **parallelogram** is a quadrilateral with opposite sides parallel, two and two.

A **rectangle** is a right-angled parallelogram.

A **square** is an equal-sided rectangle.

A **trapezium** is a quadrilateral with two sides parallel.



Definition 20.—**Polygons** are closed plane figures bounded by straight lines, and are named from the number of sides and angles.

	Sides and angles.		Sides and angles.
Triangle	3	Octagon	8
Quadrilateral	4	Nonagon	9
Pentagon	5	Decagon	10
Hexagon	6	Dodecagon	12
Heptagon	7	Quindecagon	15

A **regular polygon** has all its sides equal, and all its angles equal.

Definition 21.—A **diagonal of a polygon** is a join of two vertices, not a side of the polygon.

* No three collinear. (This is always understood in definitions of polygons.)

EXAMPLES—XII.

1. Construct a rhombus, side $1\frac{1}{4}''$, one angle 60°. Measure the shorter diagonal.

2. Construct a parallelogram, sides $1''$, $1\frac{1}{2}''$, angle 49°. Measure the shorter altitude.

3. Construct a rectangle, sides $\frac{3}{4}''$, $1\frac{3}{4}''$. Measure diagonals.

4. Construct a square of $1\frac{1}{4}''$ side. Measure diagonal.

5. Construct a trapezium, pentagon, hexagon, heptagon.

6. Name in order the five figures given in Definition 19.

Symmetry and Congruence.

Definition 22.—A figure is **symmetrical** about a line when it can be reversed about that line so as to coincide with its original position.

EXAMPLES—XIII.*

1. A limited straight line is symmetrical about its right bisector.

2. An angle is symmetrical about its bisector.

3. A circle is symmetrical about any diameter.

4. The figure formed by two circles is symmetrical about the line of their centres.

Definition 23.—Two figures are **congruent** when one can be placed on the other so as to coincide with it.

EXAMPLES—XIV.

1. Two planes or two sides of a plane, considered as infinite, are congruent.

2. Two right angles are congruent.

3. The two parts of a bisected line, or a bisected angle, are congruent.

4. Two straight lines of the same length are congruent.

5. Two equal angles, or the two faces of an angle, are congruent.

6. Two circles with equal radii are congruent.

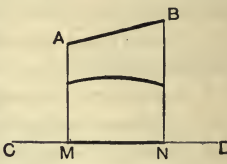
7. If two triangles ABC, DEF are congruent, in the order of the letters, which sides and which angles are equal?

8. Two rectangles of equal sides—e.g. rectangular post-cards—are congruent. So also two squares of the same side.

* Tracing-paper may be used.

PROJECTION.*

Definition 24.—The **projection** (or right projection†) of a line or curve AB on a straight line CD is the length MN between perpendiculars to CD from the *ends* of AB.



It is evident that if two lines are bounded by the same perpendiculars to a given line, their right projections on it are equal.

EXAMPLES—XV.

1. Draw a parallelogram, sides 1″, $1\frac{1}{2}$″, angle 40°. Draw and measure the projection of the first side on the opposite side, and on the adjacent side. Which is greater?

2. Draw an equilateral triangle, 1″ side. Measure the projections of two sides on the third.

3. Draw an isosceles triangle. Verify that the projections of the equal sides on the third side are equal.

AREA.

Triangles, polygons, circles, &c. enclose a certain amount of the surface of a plane, which is not the same for all figures—e.g. the floor spaces of two rooms may differ greatly in size.

Definition 25.—The **area** of a closed figure is the amount of the surface which it encloses.

The unit of area is a square, generally on unit length—e.g. square mile, square foot; and the area of a figure is said to be so many square miles, square feet, &c.

'Congruent figures have equal areas.'

Use a division of squared paper for 1 foot. Draw a plan of a room 18 ft. by 15 ft., and find its area. How can you calculate it without the plan?

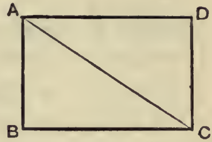
Draw several rectangles on squared paper. Verify that their area (in terms of a square of the paper as unit) is the product of lengths of their sides, or base × altitude.

Ex. How many square feet of carpet cover a floor 28 ft. by 18? If the carpet is 2 ft. wide, how many strips are wanted?

* The remainder of this chapter may be postponed if thought advisable.

† Generally called orthogonal projection.

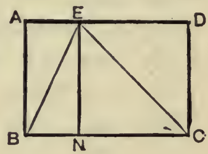
TRIANGLE—PARALLELOGRAM.



Take a rectangular post-card **ABCD**; cut in two along a diagonal **AC**.

Verify that the triangles are congruent.

What fraction of the whole area is that of each triangle?

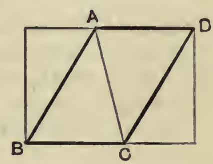


Take another card; take any point **E** in **AD**; draw **EN** perpendicular to **BC**, dividing the card into two rectangles.

Join **EB, EC.**

What fraction of area **AN** is **EBN**? of **DN** is **ENC**? of **AC** is **EBC**?

'The area of a triangle is half the product of base and altitude ($\frac{1}{2}$ base × alt.).'



Cut out a parallelogram from a post-card, as **ABCD**; cut it across a diagonal **AC**.

Verify that trs. **ABC, CDA** are congruent.

What fraction of the parallelogram is either triangle?

What is the area of the parallelogram?

'The area of a parallelogram is the product base × altitude, and is that of a rectangle of the same base and altitude.'

Draw two parallelograms of the same base and altitude, but with different angles. Show that they are equal in area. Repeat for two triangles.

EXAMPLES—XVI.

1. Move the triangle **DEC** (in the second figure above) to the left, making **DC** coincide with **AB**. The area of the whole figure is unaltered. What is the new figure? What can you conclude?

2. Draw a parallelogram, sides 2″, 3″, angle 30°. Measure its area.

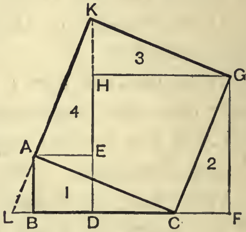
3. Draw an equilateral triangle, $1\frac{1}{2}$″ side. Measure its area.

4. Calculate the areas: triangle, parallelogram, base $1\frac{3}{4}$″, alt. 2″; trapezium, parl. sides 1″, $1\frac{1}{2}$″, alt. $1\frac{1}{4}$″. (Divide into triangles.)

5. Draw a square of 2″ side. Calculate and measure its area.

Squares of Right Triangle.

Theorem 10.*—'The square on the hypotenuse of a right triangle is equal to the sum of squares on the other sides.' (Pythagoras' theorem.)



If B is the rt. ang. of tr. ABC, make sq. ABDE, and CF eql. to BD;
∴ DF = BC.

Make sq. DFGH, and HK eql. to AB;
∴ EK = DH = DF = BC.

Join as in figure.

Then trs. 1, 2, 3, 4 are congruent.
∴ if 1 and 2 are taken from sqq. ABDE, DFGH, and 3 and 4 added, the area is unchanged.

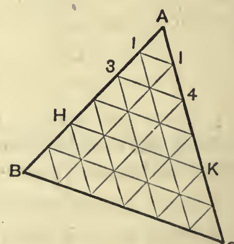
But the figure is then ACGK, the sq. on AC;
∴ sq. on hyp. AC = sum of sqq. on sides AB, BC.

Note. ACGK can be shown to be a square thus:
Produce KA, CB to meet in L;
then ang. L = KAE (same parts) = GCF,
∴ KA ∥ GC; similarly KG ∥ AC.

Also ang. KAC = KAE + EAC = BAC + EAC = a right angle;
∴ fig. ACGK is an equal-sided right-angled parallelogram;
i.e. ACGK is a square.

Areas of Equiangular or Similar Triangles.

Divide a triangle ABC into small triangles by parls. dividing the sides into, say, 6 equal parts.



Count up from A, and write down
(i.) on the left, the number of triangles in successive trapeziums;
(ii.) on the right, the total number of triangles in successive triangles, as AHK.

What do you notice about the two sets of numbers?

Give the ratio of areas ABC : AHK. Compare with AB : AH.

* This is meant as an exercise in drawing; the figure may be drawn on a post-card and cut out.

Fold a piece of paper round a coin or a cylinder, and mark the length to fold just round. Measure this length and the diameter, and calculate the ratio $\frac{\text{circumference}}{\text{diameter}}$. Thus:

	Cycle Wheel.*	Cylinder.*	Collar-box.*
Circumf.	7 ft. $3\frac{1}{2}''$	21·46 cm.	$23\frac{7}{32}''$
Diam.	2 " $3\frac{7}{8}''$	6·83 cm.	$7\frac{13}{32}''$
Ratio π	3·139...	3·142...	3·135...

Thus the ratio $\frac{\text{circumference}}{\text{diameter}}$, denoted by π, is a little greater than 3. It has been calculated to several hundred decimal places, and its value to 5 places is 3·14159. We may use 3·14 as an approximation.

Area of a Circle.

If a circle is divided into very small sectors, each sector is practically a triangle, and its area is half the product of base (arc of sector) and altitude (radius).†

Hence the area of a circle is equal to that of a triangle whose alt. is the radius and base the circumference of the circle.†

The length l, and area A of a circle, radius r, are

$$l = 2\pi r,\ \mathsf{A} = \pi r^2,\ \text{where } \pi = 3{\cdot}14\ldots$$

The arc or area of a sector can be calculated, when its angle is known, as a fraction of that of the whole circle.

EXAMPLES—XVII.

1. Write down the area of the second circle above.

2. Calculate lengths and areas of circles, radii 1·3″, 2·7 cm., $5\frac{1}{2}$ yd. (the last area in square poles also).

3. Calculate arcs and areas of sectors of a circle, radius 2·4 cm., angles 45°, 60°, 90°.

4. Measure π for a bicycle wheel or a draughts-man.

5. The diameter of a carriage wheel is 44″. How many turns does it make in a mile?

* These were measured with a tape. A set of draughts-men, unpolished, serves for a class. The circumference can be measured by rolling on a diagonal scale.

† This is strictly true, though the proof is incomplete. See Chapter V.

EXAMPLES—XVIII.

1. Draw a triangle, $a=2{\cdot}4$ cm., $b=2{\cdot}7$ cm., $c=3{\cdot}3$ cm., and measure the greatest angle.

2. Draw a triangle, $\mathsf{A}=54°$, $\mathsf{B}=65°$, $c=1{\cdot}8''$. Measure the altitude from B to AC.

3. Draw a triangle, $a=1{\cdot}5''$, $b=1{\cdot}2''$, $\mathsf{C}=53°$, and measure the side c.

4. Two posts on one side of a river are 100 yd. apart. A post on the opposite bank is 60 yd. from one post and 80 yd. from the other. Draw a plan, 1 cm. to 10 yd., and find the width of the river.

5. A rope of a giant's stride 16 ft. high just reaches the ground. When held by a boy 10 ft. from the post, it just reaches the top of his head. Find his height. (Use 1 cm. for 2 ft.)

6. A man lying on the ground 30 ft. from a house can just see the roof. If the slope of the roof is 50° with the ground, find the height of the house wall. (Use 1″ to 10 ft.)

7. Prick two points A, B on paper 5 cm. apart. Without pen or pencil show the straight line AB, and bisect it.

8. Draw a diagram to show the four cardinal points (N., S., E., W.). Show also the directions N.E., N.W., S.E., S.W.

9. Write down the values of the angles formed by the following directions: N., S.E.; E., N.E.; E., S.W.; S., N.E.

10. (Range finding.) The directions of a battery C make angles $\mathsf{A}=90°$, $\mathsf{B}=79°$ with a line AB, 1000 yd. long. Find the distance of the battery from A. (Use 1″ to 1000 yd.)

11. At a distance of 180 ft. from its foot, the direction of the top of a tower makes with the ground an angle of 30° (called its angle of elevation, or its elevation). Find the height of the tower.

12. A man walks 5 miles N., 4 miles N.W., 5 miles S., and then home. Draw a plan, and state the whole length of the walk. What is the figure of the walk?

13. A man is 2 miles N. of another. They start walking at the same time in the same direction, N.W., at 3 and 4 miles an hour respectively. How far apart, and in what direction from each other, are they at the end of 2 hrs. 50 min.?

14. Draw an isosceles triangle, $a=b=2\frac{1}{2}''$, $c=2''$. Measure the angle C.

15. Draw an isosceles triangle, $a=b=4$ cm., $\mathsf{C}=73°$. Measure the third side.

16. Draw a rhombus, side $1\frac{1}{4}''$, angle 67°. Draw an altitude.

17. Draw a parallelogram, sides $\frac{7}{8}''$, $1\frac{5}{8}''$, angle 79°.

18. Draw a parallelogram, sides 3·2 cm., 2·7 cm., one diagonal 4·5 cm.

19. Draw a circle, radius 1″, centre **O**; draw any radius **OA**. With centre **A**, radius 1″, draw a circle, cutting the first circle in **B**. Complete the triangle **ABO**. What kind of triangle is **AOB**? What angle is **AOB**? How many such triangles will go just round the first circle?

20. Draw a circle, radius 3 cm. Step off points **A, B, C, D, E, F** round the circle at equal distances of 3 cm.; join the points in order. What kind of hexagon is **ABCDEF**? What do you know about its sides? its angles?

21. Draw a circle of 1″ radius. Mark off 6 points 1″ apart on the circle; with these points as centres, radius ½″, draw six circles. With the original centre, radius 1½″, draw a circle to touch all these.

22. Make an angle of 72° at the centre of a circle 3 cm. radius. What fraction of the whole circumference does it cut out? Complete a regular pentagon in the circle. Measure its side.

23. Draw a circle, radius 1·4″. Draw two diameters at right angles. Join their ends in order. What figure is formed? Measure its side.

24. How can you use Ex. 23 to draw a square when its diagonal is given? Draw a square of diagonal 7 cm. Measure its side.

25. Draw a circle, 1″ radius. Set off three radii at angles of 120°, and mark points **A, B, C** on them ½″ from the centre. With centres **A, B, C**, radii ½″, draw three circles. With the original centre draw a circle through the points where the three circles cut each other.

26. Divide a circle into eight equal sectors. What figure is formed by joining the ends of the radii in order?

27. If the angle of vision of an eye is 120°, what width of country can the eye take in at a glance at a distance of 5 miles? (Use 1 cm. to the mile.)

28. Draw a regular dodecagon (twelve-sided polygon) in a circle, radius 2″. (Make angle of 30° at the centre.)

29. Fold over a sheet of paper, and mark the fold. Make a second fold perpendicular to the first, cut across to make a fourfold triangle, and unfold. What is the figure? What can you infer about its diagonals?

30. Repeat Ex. 29, but cut across so as to make the fourfold triangle isosceles. What is the figure now? And what additional fact do you know about its diagonals?

31. Describe a parallelogram, sides 2·7 cm., 3·8 cm., angle 115°. Draw perpendiculars to one side (prolonged sufficiently) from the ends of the opposite side. State two facts about them.

32. Take a straight line **AB**, 4·8 cm. long. Draw circles, centres **A, B**, radii 3 cm., to cut in **C, D**. What figure is **ACBD**? What line is **CD** with reference to **AB**?

33. Draw squares on sides of 2·3 cm. and 4·6 cm. Measure their diagonals. How many of the first are contained in the second?

34. Draw a square on diagonal 5·6 cm. Measure its side. (See Ex. 23.)

35. Draw a rectangle, one side 2·7 cm., diagonal 4·3 cm. Measure the other side, and the other diagonal.

36. Draw a right triangle, hyp. $3\frac{1}{4}''$, one side $3''$. Measure the other side. Does your construction resemble that of Ex. 35? Why?

37. Draw a parallelogram, one side $1''$, one angle 136°, and the diagonal through this angle $1\frac{1}{4}''$. How many can you draw?

38. Draw a straight line **AB**, $1{\cdot}7''$. At **A**, **B** make perpendiculars **AC**, **BD** $\frac{3}{4}''$ long. State an additional fact about **AC**, **BD**. What kind of figure is **ACDB**? State two facts about **CD**, **AB**.

39. Draw two parallel lines $1''$ apart. On one of them mark off **AB** $1\frac{1}{2}''$. Draw **AC** to the other $1\frac{1}{4}''$. Make **CD** on the second parallel $1''$. Join **BD**. What kind of figure is **ACDB**?

40. Draw a trapezium as in Ex. 39. Bisect the two non-parallel sides **AC**, **DB** at **E**, **F**. Measure **AB**, **CD**, **EF**; compare 2.**EF** with the sum **AB**+**CD**.

41. Draw a parallelogram, sides $1{\cdot}4''$, $2{\cdot}3''$, one diagonal $3''$. Compare its opposite sides and opposite angles. Can you prove opposite angles equal? (Produce sides and mark angles of same parts.)

42. Draw a parallelogram, sides 3·2 cm., 4·7 cm., angle 137°. Write in each of the other angles its number of degrees.

43. Make a parallelogram on squared paper, base 7 divisions, altitude 5, one angle 60°. Count the number of squares in it, and compare with a rectangle of sides 7 divisions and 5.

44. Make two squares on squared paper, sides 3 divisions and 7 divisions. Write down the ratio or fraction of their areas. Compare with that of their sides.

45. Draw a triangle, $a=2{\cdot}3$ cm., $b=3{\cdot}2$ cm., $c=2{\cdot}9$ cm. Draw a line parallel to **BC**, bisecting **AC** and **AB**; show that the triangle cut off is a quarter of the whole. (Draw parls. to the sides as on p. 36.)

46. Calculate the areas: rhombus, base $1{\cdot}7''$, alt. $2{\cdot}3''$; rectangle, sides 132 yd., 79 yd.; triangle, base 35 ft., alt. 27 ft.

47. One side of a rectangular field is 72 yd., and makes an angle of 30° with a diagonal. Draw a plan and calculate the area. (1 cm. to 10 yd.)

48. Calculate, without drawing, the length of the hypotenuse of the right triangle whose sides are (i.) 2·7 cm., 3·9 cm.; (ii.) $3\frac{3}{4}''$, $2\frac{1}{4}''$; (iii.) 3 miles, 4 miles; (iv.) 127 yd., 322 yd.

49. Calculate, without measuring, the altitude of an equilateral triangle of $2''$ side. Deduce from it those of equilateral triangles of $1''$, 3 cm., 5 ft., 7 yd. side.

50. The distance of two park gates cannot be measured because of a lake; but a third gate is 500 yd. from one, and 300 yd. from the other in a line perpendicular to the join of the two gates. Find their distance.

51. A street is 100 ft. wide. Calculate and measure the length of a string stretched from the bottom of a wall on one side to the top of a 60 ft. wall on the other.

52. Draw two squares, sides 2·3 cm. and 6·9 cm. How many of the former are contained in the latter? Why?

53. Draw a triangle ABC, $a=2{\cdot}3''$, $b=1{\cdot}9''$, $c=3{\cdot}1''$. Divide the side AB into 5 equal parts; draw parallels to the sides as on p. 36. How many small triangles are there? Make AH, AK $\frac{3}{5}$ of AB, AC. Write down the ratio or fraction of the areas AHK, ABC. Compare with that of their sides.

54. How far does each of the following wheels go in one revolution: (i.) driving wheel, 6 ft. diam.; (ii.) cycle wheel, 28″ diam.; (iii.) carriage wheel, 3 ft. 6 in. diam.?

55. How many revolutions do the wheels of Ex. 54 make in a mile?

56. Calculate the area of each wheel in Ex. 54.

57. On what area can a donkey feed when tethered by a 20 ft. rope?

58. Calculate the areas of sectors of a circle of 1″ radius, angles 30°, 45°, 60°, 72°, 120°.

59. Draw a circle, 1″ radius. Draw two diameters at right angles, and form a square by joining their ends. What is the area of the square? of the part of the circle left after cutting out the square?

60. Draw a square ABCD, side 4 cm. Draw diagonals AC, BD; and draw perpendiculars to these at the points A, B, C, D. What is the new figure? Find its area.

61. Draw a circle, 3 cm. radius. Draw two diameters AB, CD at right angles, and perpendiculars to these at A, B, C, D. What is the new figure? its area?

62. Find the area left from the outside square in Ex. 61 after the circle has been cut out.

63. Draw a circle, 1″ radius. Mark off 6 points 1″ apart on the circle, and join three alternate (every other) points. What figure is formed? Measure its side and altitude, and calculate its area.

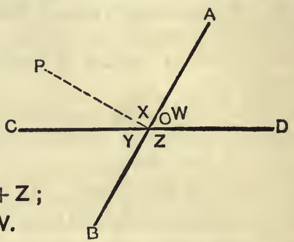
64. Can you calculate the altitude of the triangle of Ex. 63 without measuring it? What is its calculated value?

CHAPTER II.

ANGLES—PARALLELS—TRIANGLE—CIRCLE.

Theorem 11.—'Two straight lines which meet have any two adjacent angles supplements, and any two opposite angles equal.' *

If AB, CD form at O the four angles X, Y, Z, W, and OP ⊥ AB;
then (i.), X + Y = POA + POB
= 2 rt. angs.;
∴ X = 2 rt. angs. − Y
= suppt. of adj. ang. Y.

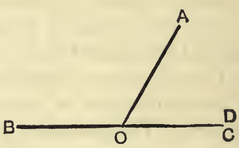


Similarly, Y = suppt. of Z,
Z of W, W of X.

Also (ii.), X + Y = 2 rt. angs. = Y + Z;
∴ X = opp. ang. Z; similarly, Y = W.

Theorem 12.—'Two adjacent angles at a point which are supplements and have a common side, have their other sides in a straight line.'

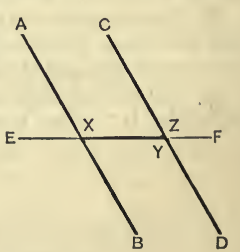
If adj. angs. AOB, AOC are suppts.
and have side AO common,
produce BO to D;
∴ ang. DOA = suppt. of BOA = COA;
∴ CO coincides with DO;
i.e. BO, CO are in a straight line.



Theorem 13.—'Two parallels make equal alternate interior angles with any third line; and two straight lines making equal alternate angles with a third are parallel.'

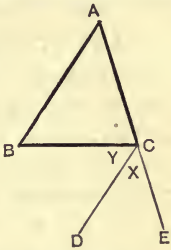
If AB, CD make with EF alt. angs.
X, Y and angs. X, Z towards same parts;
then (i.), if AB ∥ CD,
X = Z (same parts)
= Y, the alt. ang.

Also (ii.), if X = alt. ang. Y,
then Z = Y (opp. ang.)
= X, towards same parts;
∴ AB ∥ CD.



* This should be proved practically by the protractor.

Theorem 14.—'The sum of the three angles of a triangle is two right angles, and an exterior angle is equal to the sum of the two interior opposite angles.'

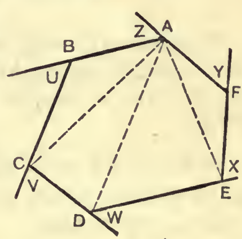


If ABC is a triangle, and CD parl. to AB makes angles, Y with BC,
and X with AC produced;
then ang. $A = X$, same parts,
and $B = Y$, alt. ang.,
$\therefore$ $A + B + C = X + Y + C = 2$ rt. angs.;
and ext. ang. $BCE = X + Y = A + B$, int. opp. angs.

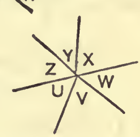
Cor. (i.).—'An exterior angle of a triangle is greater than an interior opposite angle.'

Cor. (ii.).—'If one angle of a triangle is right or obtuse, each of the others is acute.'

Theorem 15.—'The sum of the angles of a polygon is twice as many right angles as the number, less two, of the sides.'



If one vertex A of the poln. ABCDEF is joined to the others, the first and last triangles have each two sides of the poln., and the rest one each;
$\therefore$ there are as many triangles as the number, less two, of the sides;
$\therefore$ sum of angs. of poln.
= sum of angs. of triangles
= twice as many rt. angs. as the number, less two, of the sides.



Note. If there are n sides,
sum of angs. $= (2n - 4)$ rt. angs.

Cor (i.).—'The sum of angles of a quadrilateral is 4 rt. angles.'

Cor. (ii.).—'The sum of exterior angles of a polygon is 4 rt. angles.' *

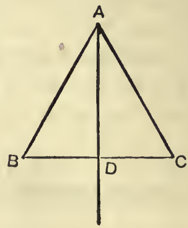
If parallels are drawn at a point to the sides of poln.,
sum of ext. angs. $U + V + \ldots = 4$ rt. angs.

* **If there are any re-entrant angles, this is not true without modification.**

Theorem 16.—'**In an isosceles triangle, the angles opposite the equal sides are equal; and the bisector of angle of the equal sides is the right bisector of the third side.**'

If AB = AC in isosceles tr. ABC,
and AD bisects ang. A,
fold over the part DAC on to DAB;
then AC comes on AB, ∵ ang. DAC = DAB;
and point C comes on B, ∵ AC = AB;
∴ ang. C coincides with B;
i.e. ang. C = B.

Also DC coincides with and = DB,
∴ D is mid point of BC;
ang. ADC coincides with and = ADB,
∴ AD ⊥ BC;
∴ AD is the right bisector of BC.

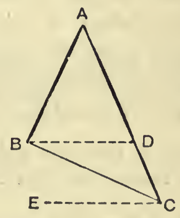


Cor.—'**An isosceles triangle is symmetrical about the bisector of angle of the equal sides.**' *

Theorem 17.—'**In any triangle one angle is greater than, less than, or equal to another, according as the opposite side of the first is greater than, less than, or equal to that of the other.**'

If AC > AB in tr. ABC,
make AD eql. to AB, and CE parl. to BD;
then ang. B > ABD
> ADB, ∵ AB = AD,
> ACE, same parts,
> C;
i.e. (i.), if AC > AB, ang. B > C.

Similarly (ii.), if AC < AB, B < C;
and (iii.) if AC = AB, B = C (Th. 16).

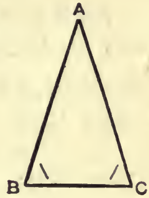


Cor.—'**The hypotenuse is the longest side of a right triangle.**'

Ex. If one angle of a triangle > another, is the side opp. the greater angle greater or less than that opp. the less angle? Why?

* This is a very important principle.

Theorem 18.—'If two angles of a triangle are equal, the sides opposite the equal angles are equal.'



If the angle B = C in tr. ABC, reverse the triangle so that points B, C are interchanged; then the angles B, C are interchanged;

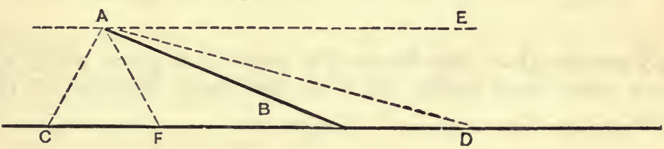
∴ the point A has its original position,

∴ CA, BA are interchanged.

∴ CA = BA.

Ex. Prove this theorem by the previous theorem.

Theorem 19.—'Any two non-parallel lines in a plane meet if produced far enough.' *



If AB, CD are non-parallel,
make AE parl. to CD, and CF eql. to CA;

∴ ang. CAF = CFA

= alt. ang. FAE.

∴ AF bisects ang. CAE.

Hence if any line AC through A meets CD, the bisector AF of ang. CAE also meets CD.

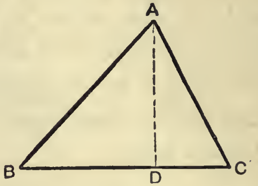
Similarly, the bisector of FAE meets CD, and every successive bisector, as long as the process is continued, meets CD.

But if the process is sufficiently continued, a bisector AD is obtained making the ang. DAE less than BAE.

∴ AB lies inside the triangle CAD, and therefore meets CD if produced far enough.

* This demonstration may be postponed by beginners. The result should be learnt, being important for constructions.

Theorem 20.—'Any two sides of a triangle are together greater than the third.'



If AD is perp.* to BC in tr. ABC;
then hyp. AB $>$ BD, in tr. ABD.

Similarly, AC $>$ DC;

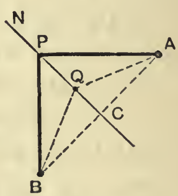
$\therefore$ AB + AC $>$ BD + DC
$>$ BC;

i.e. $b+c>a$;
similarly, $a+b>c$, $c+a>b$.

Cor.—'Any side of a triangle is greater than the difference of the other two.'

Ex. Draw the figures and prove for right or obtuse triangles.

Theorem 21.—'The locus of a point in a plane equidistant from two fixed points in it, is the right bisector of their join.'



If P is any point equidistant from the fixed points A, B,
then PA = PB in tr. APB;

$\therefore$ NC bisecting ang. APB is the right bisector of AB (Th. 16);

i.e. every point P on the locus is on the right bisector of AB.

Also, if Q is any point on the right bisector CN, the figure AQB is symmetrical about CN;

$\therefore$ QA = QB;

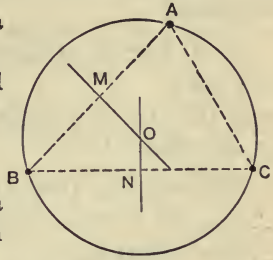
i.e. every point of the right bisector CN is on the locus.

$\therefore$ the right bisector of AB is the locus.

Ex. Construct a right bisector and an isosceles triangle in paper without ruler or compass.

*By definition a perp. to BC exists at B; a parl. to this through A meets BC by Th. 19, and is perp. to BC.

Theorem 22.—'One only circle can be drawn through any three points not in a straight line.'

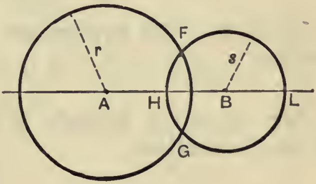


If A, B, C are three points not in a straight line,
the right bisectors MO, NO of AB and CB are non-parallel,
because AB, BC are non-parallel;
∴ MO, NO meet in a point O;
∴ O is equidistant from A, B, C, and a circle, centre O, rad. OA, passes through A, B, C.

Also, the centre of a circle through A, B, C is equidistant from A, B and lies in MO, and similarly in NO;
∴ O is the only centre, and there is one only circle.

Cor.—'Two circles cannot meet in more than two points.'

Theorem 23.—'Two circles cut, in two only points, when the join of their centres is greater than the difference, and less than the sum, of their radii.' *



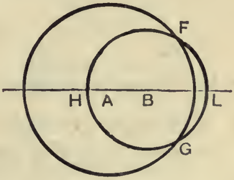
If A, B are centres of circs., rad. r, s,
$r \geqq s$, and AB meets the circle s in H, L;
then if AB $> r - s$ and $< r + s$,

(i.) $\text{AL} = \text{AB} + \text{BL}$
$> r - s + s$
$> r,$

∴ L is outside circle r.

(ii.) $\text{AH} + \text{BH} < r + s$,
∴ $\text{AH} < r$,

∴ H is inside circle r;
∴ the circle s is partly outside and partly inside the circle r, and cuts it in two only points F, G.

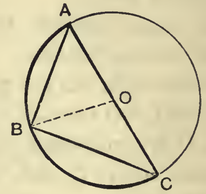


Also, by symmetry about AB,

Cor.—'The line of centres of two intersecting circles is the right bisector of the common chord.'

* Beginners may use the first figure only.

Theorem 24.—'The angle in a semicircle is a right angle; and the circumcircle of a right triangle has the hypotenuse as diameter.'



(i.) If ABC is a semicircle, centre O, join OB;

then OA = OB in tr. AOB,

∴ ang. OBA = A.

Similarly, ang. OBC = C,

∴ ang. B = A + C in tr. ABC;

∴ B is a right angle.

(ii.) If ABC is a rt. triangle, B the rt. ang., make ang. ABO eql. to A;

∴ OB = OA.

Also, ang. OBC = compt. of OBA = compt. of A

= OCB, ∵ B is a rt. ang.;

∴ OB = OC.

∴ The circle, centre O, diameter AC, is the circumcircle of rt. triangle ABC.

Cor. (i.).—'The hypotenuse of a right triangle is double its median.'

Cor. (ii.).—'A triangle is right when one of the sides is double its median.'

EXAMPLES—XIX.

1. Show that angle BOC = 2 . BAC.

2. Any angle at the centre of a circle—i.e. having two radii as its sides—is double the angle at the circumference on the same arc.

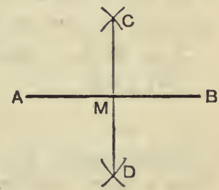
3. Ang. BAC = compt. of BCA.

4. The right bisector of a chord of a circle bisects either arc of the chord. (Use symmetry.)

5. The diagonals of a rectangle are equal and bisect each other.

Bisectors, Circumcircle.

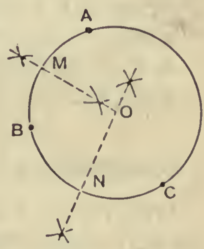
Construction 1.—'Draw the right bisector of a given line AB; or, bisect a given line at right angles.'



With centres A, B, any radius greater than half AB,
draw equal circles cutting in C, D;
join CD, cutting AB in M.
C, D, equidistant from A, B, lie in the right bisector of AB;
∴ CD is rt. bisector of AB, and M the mid point.

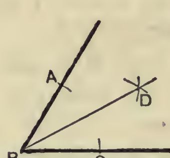
Ex. Show also that AB is rt. bisector of CD. What figure is the quadrilateral ACBD?

Construction 2.—'Draw the circle through three points A, B, C, not in a straight line.'



Draw MO, NO rt. bisectors of AB, BC;
∴ O is equidistant from A, B, C.
Draw circle, centre O, radius OA, passing through A, B, C.
(This circle is the **circumcircle** of tr. ABC.)

Construction 3.—'Draw the bisector of a given angle.'



If B is the angle, draw arc AC, centre B, any radius, meeting the sides in A, C.
With centres A, C, same * radius, draw arcs D, and join BD.
Then tr. ABC is isosceles;
∴ bisector of ang. B is rt. bisr. of AC, and therefore traverses D;
∴ BD is bisector of ang. B.

Note. If AB, CB are produced to cross, BD produced bisects the opp. ang. at B, and a perp. to BD at B bisects the two adjacent angles.

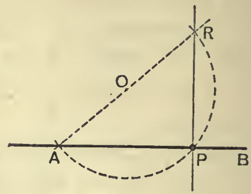
Ex. Show that the bisectors of adjacent angles of two lines are perpendicular.

* This is convenient, though not necessary.

Construction 4.—'Draw a perpendicular to a straight line at any point in it, by ruler and compass.' *

If P is the point in AB,
with any point O as centre, draw arc APR,
join AO to meet arc in R, join RP.

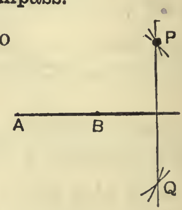
Ang. RPA, in semicircle, is right;
$\therefore$ PR $\perp$ AB.



Construction 5.—'Draw a perpendicular to a straight line from a point not on the line, by ruler and compass.' *

If P is the point, AB the line, with any two points A, B on the line as centres, draw arcs PQ, meeting in Q; join PQ.

A, B are on right bisector of PQ;
$\therefore$ PQ $\perp$ AB.



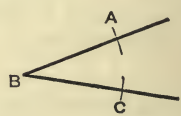
Construction 6.—'Draw an angle equal to a given angle.'

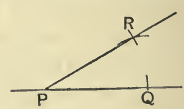
With the point B of the given angle as centre, any radius, draw arc AC cutting its sides in A, C.

With centre P in any line PQ, same radius, draw arc QR.

With centre Q, radius CA, draw arc R; join PR.

If ang. B is placed on P so that BC coincides with PQ, A comes on each of the arcs at R, and coincides with R;
$\therefore$ ang. P = ang. B.



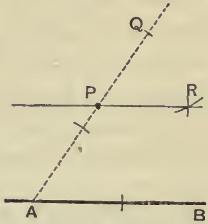


Scale of Chords.—This is formed by setting off the chords of 1°, 2°, 3°... at the centre of a given circle along a straight line from a fixed point O.

To construct an angle, say 32°, from it, draw an arc QR, centre P, radius chd. of 60°; draw arc R, centre Q, rad. chd. of 32°;
$\therefore$ ang. P = 32°.

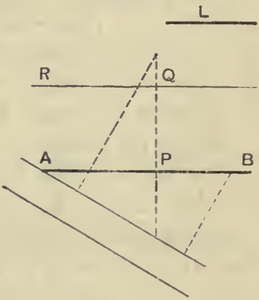
* Of theoretical interest only. Construction by set-square is simpler.

Construction 7.—**'Draw a parallel to a given line from a given point, by ruler and compass.'** *



Draw any str. line QA through the given point P, meeting the given line AB in A; make ang. QPR eql. to PAB;
∴ PR || AB.

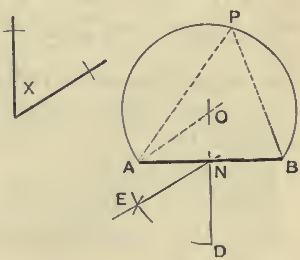
Construction 8.—**'Draw a parallel to a given straight line at a given distance from it.'**



Set the hyp. of set-square along the given line AB, fix a str.-edge along one side, and interchange the sides of set-square, so that hyp. comes into posn. PQ, perp. to AB.

Prick off PQ from AB, along hyp., eql. to given line L;
draw QR parl. to AB, at given dist. L.

Construction 9.—**'On a given straight line draw an arc of a circle to contain a given angle.'**



If AB is the line,
draw right bisector DNO,
make ang. DNE eql. to given ang. X,
draw AO parl. to EN.

With centre O, draw arc APB.

Ang. APB = $\frac{1}{2}$AOB (at centre) †
= AON = DNE = X;

∴ arc APB contains given angle X.

Note. If X is given in degrees, the angle NAO may be drawn at once as the *complement* of the given angle X.

* Of theoretical interest only.

† See Ex. 2, Th. 24.

Construction 10.—'Triangles.'

(Three conditions, one at least of which is not an angle, determine a triangle: there are sometimes alternative solutions.)

State the number of solutions in each case.

Construct a triangle, having given

(i.) three sides a, b, c;

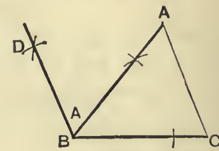
(ii.) two sides and their angle, a, b, C;

(iii.) two sides and an angle opposite one, a, b, A;

(iv.) one side and two adjacent angles, a, B, C;

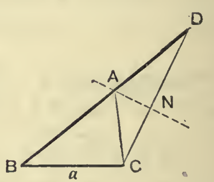
(v.) one side, one adjacent, and the opposite angle, a, B, A.

If A, B are given in degrees, calculate C, and use (iv.). Otherwise:



Make $BC=a$, $CBA=B$, $ABD=A$;
make CA parl. to BD;
∴ ang. $CAB=$ alt. ang. $ABD=A$;
∴ ABC is the triangle.

(vi.) One side, one adjacent angle, and sum of the other (or of the three) sides, a, B, $b+c$.*



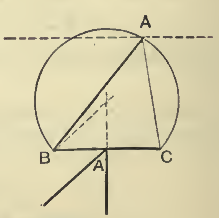
Make $BD=b+c$, $DBC=B$, $BC=a$;
draw NA, rt. bisector of CD, join AC;
∴ $AC=AD$, and $AB+AC=BD=b+c$;
∴ ABC is the triangle.

(vii.) One side, the opposite angle, and sum of the other sides, a, A, $b+c$.*

Make $BD=b+c$, $BDC=\frac{A}{2}$, $BC=a$;
draw NA, rt. bisector of CD, join AC.

Ang. $BAC=ACD+ADC=2ADC=A$,
and $AC=AD$;
∴ $BA+AC=b+c$;
∴ ABC is the triangle.

(viii.) One side, its altitude, and opposite angle, a, h, A.



Make $BC=a$, arc BAC containing A.
Draw a parl. to BC at distance h,
cutting the circle in A;
∴ ABC is the triangle.

* Similar constructions serve when $b-c$ is given instead of $b+c$.

EXAMPLES—XX.

THEOREMS.

1. When two lines meet, the bisectors of adjacent angles are perpendicular.

2. If two straight lines, **AB**, **CD**, on the same side of **AC** make the interior angles **A**, **C** supplements, then **AB** $\parallel$ **CD**.

3. If **AB** $\parallel$ **CD** in trapezium **ABCD**, and **AE** is drawn making angle **BAE** equal to **CDA** towards the same parts, show that **AE**, **AD** are in a straight line.

4. If **AB** $\parallel$ **CD**, the angle **BAD** is either the supplement of **ADC** or equal to it.

5. If two straight lines make two interior angles on the same side of a third line supplements, the two first lines are parallel.

6. If two lines a, b are parallel respectively to c, d, the angles ab and cd towards the same parts are equal.

7. One only perpendicular can be drawn from a given point to a given line.

8. If **ABCD** is a trapezium, and **AB** $\parallel$ **CD**, ang. **A** = suppt. of **D**.

9. In an isosceles triangle a parallel to the unequal side cuts off equal parts from the equal sides.

10. If a straight line cuts off equal parts from the equal sides of an isosceles triangle, it is parallel to the third side.

11. The bisectors of the equal angles of an isosceles triangle form an isosceles triangle with the unequal side.

12. Show that equilateral triangles have all their angles equal. What is the value of each?

13. If any angle of an isosceles triangle is 60°, the triangle is equilateral.

14. The exterior angle of the equal sides of an isosceles triangle is equal to twice one of the equal angles.

15. If the straight line bisecting an exterior angle of a triangle is parallel to the opposite side, the triangle is isosceles.

16. If a bisector of angle of a triangle is perpendicular to the opposite side, the triangle is isosceles.

17. If **BD**, **CE** bisecting the equal angles **B**, **C** of a triangle meet **AC**, **AB** in **D**, **E**, then **BE** = **CD**.

18. Show also in Ex. 17 that **DE** = **BE** and $\parallel$ **BC**.

19. If **D**, **E** are points on the equal sides **AC**, **AB** of an isosceles triangle such that **BE** = **CD**, then **DE** $\parallel$ **BC**.

20. If in Ex. 19, DE also = BE, then BD bisects angle B.

21. The perpendicular is the shortest straight line from a given point to a given straight line.

22. If lines OP, OQ, &c. are drawn from a point O to points P, Q, &c., in a straight line, they increase with their angular distance from the perpendicular ON.

23. Any line from a vertex of a triangle to the opposite side is less than the greater of the two other sides.

24. If P is a point inside a triangle ABC, then BP + PC is less than BA + AC, and angle BPC is greater than BAC.

25. Show also that PA + PB + PC in Ex. 24 is less than the sum, and greater than half the sum, of the sides.

26. The sides of a quadrilateral are together greater than the sum of the diagonals.

27. The sum of the diagonals of a quadrilateral is greater than half the sum of the sides.

28. If two straight lines are right bisectors of each other, the joins of their ends form a rhombus.

29. If from any point on the bisector of an angle parallels are drawn to the sides, a rhombus is formed.

30. A rhombus is a parallelogram.

31. The diagonals of a rhombus bisect its angles, and bisect each other at right angles.

32. If a diagonal of a parallelogram bisects its angles, the figure is a rhombus.

33. The diagonals of a square are equal, bisect each other at right angles, bisect the angles, and make angles of 45° with the sides.

34. The right bisectors of sides of a square pass through the intersection of diagonals.

35. If AB, AC are equal sides of an isosceles triangle, and O is the circumcentre, OA bisects angle A.

36. If circles are drawn with the ends of a straight line as centres, and equal radii greater than half the line, show that they cut twice.

37. The circumcentre of an equilateral triangle is on each bisector of angle.

38. If O is the circumcentre of an equilateral triangle ABC, the angle BOC is 120°.

39. If the angle A of the equal sides of an isosceles triangle ABC is 50°, and O is the circumcentre, show that angle BOC is 100°.

40. If angles B + C in a triangle = A, the triangle is right-angled.

41. If the median AD of a triangle ABC is half BC, the triangle is right-angled.

42. In an isosceles right triangle the perpendicular and median through the right angle coincide.

43. The right bisectors of the right-angled sides of a right triangle meet on the hypotenuse.

44. If one line **AB** meets a given line **AC**, any parallel to **AB** meets **AC**.

45. If **OM**, **ON** are perpendicular respectively to two parallels, then **OM**, **ON** are in a straight line.

46. If **OP**, **OQ** make equal angles respectively, towards the same parts, with two parallels, then **OP**, **OQ** are in a straight line.

47. Straight lines bisecting two consecutive angles of a parallelogram are at right angles.

48. By producing two sides of an angle of a parallelogram, show that opposite angles of a parallelogram are equal.

49. If four straight lines at a point make opposite angles equal in pairs, they are two and two in a straight line.

50. A straight line **NDE**, perpendicular at **N** to the side **BC** of an isosceles triangle, meets the equal sides **AB**, **AC** in **D**, **E**; show that the triangle **ADE** is isosceles.

51. If a right triangle has one acute angle double the other, the hypotenuse is double the shortest side.

52. A trapezium **ABCD** has **AB** parallel to **CD**, and angle **C** equal to **D**. If **AD**, **BC** meet in **E**, show that triangle **EAB** is isosceles.

53. In a quadrilateral **ABCD**, angle **C** = **D**, side **AD** = **BC**; show that **AB** ∥ **CD**. (Produce sides to meet.)

54. On the side **BC** of a triangle as diameter a circle is drawn cutting **AB**, **AC** in **D**, **E**; show that **BDC**, **BEC** are right triangles, and have equal medians from **D**, **E**.

55. Two isosceles triangles have the angles of their equal sides equal; show that their other angles are also equal.

56. If **DE** parallel to the unequal side **BC** of an isosceles triangle cuts the equal sides in **D**, **E**, the right bisector of **BC** is also the right bisector of **DE**.

57. If the sides **AB**, **AC** of a triangle are produced to **D**, **E**, and the angle **DBC** is equal to **ECB**, the triangle is isosceles.

58. On **AB**, one of the equal sides **AB**, **AC** of an isosceles triangle, a point **D** is taken. Show that the angle **ADC** > **ACD**.

59. **MP**, **MQ** are drawn perpendicular to a given line **AB** on opposite sides of it, from a point **M** on the line; show that **MP**, **MQ** are in a straight line.

60. If also **MP** = **MQ**, in Ex. 59, what do you know about the sides of the figure **APBQ**? about its diagonal **AB**?

EXAMPLES—XXI.

CONSTRUCTIONS.

1. Calculate the sum of angles of polygons of 7, 8, 10 sides. Also the sum of exterior angles.

2. Calculate each angle of regular polygons of 5, 6, 7 sides.

3. Draw a triangle, $a=1''$, $b=1\frac{1}{4}''$, $\mathsf{C}=40°$. Bisect **C** and measure its parts.

4. Draw an isosceles triangle, $a=b=1\frac{1}{8}''$, $\mathsf{C}=46°$. Draw the right bisector of c; produce it as far as **C**.

5. Draw a right triangle, hyp. $a=3{\cdot}2$ cm., $b=1{\cdot}6$ cm. Measure **C**.

6. Draw a triangle, $a=\frac{7}{8}''$, $\mathsf{B}=52°$, $\mathsf{C}=67°$. What is the 3rd angle? Draw a perpendicular **AD** on **BC**, produce so that **DE**=**AD**. Measure angles **EBC**, **ECB** and compare with **B**, **C**.

7. Draw a triangle, $a=1\frac{3}{4}''$, $b=2\frac{1}{4}''$, $c=2\frac{1}{2}''$; bisect **B**, draw **CD** perpendicular to the bisector and produce to meet **BA** in **E**. Measure **BE**.

8. Draw any straight line, and mark two points **A**, **B** off the line; find a point **P** in the line equidistant from **A**, **B**.

9. Construct a point **P** on a given circle equidistant from two points **A**, **B** on the circle.

10. Calculate the angle of an equilateral triangle, and construct the angle.

11. Construct a triangle, one side 3·7 cm., two angles 60°. What kind of triangle is it?

12. Trisect a right angle by ruler and compass.

13. If one angle of a triangle **A** is four times each of the others, calculate the angles; construct such a triangle, $a=2{\cdot}6$ cm.

14. Draw a triangle, $a=2{\cdot}8$ cm., $b=3{\cdot}5$ cm., $c=4{\cdot}2$ cm.; construct the circumcircle and measure its radius.

15. Construct a triangle, $a=b=3{\cdot}2$ cm., altitude from **AB**, 2·4 cm.

16. Construct an isosceles triangle, $a=b$, $c=1{\cdot}8''$, $\mathsf{C}=56°$. (Calc. angs. **A**, **B**.)

17. Find the locus of mid points of chords of a circle, parallel to a given diameter.

18. Construct a triangle, $a=3{\cdot}3$ cm., $\mathsf{B}=43°$, **R** (circumradius) = 1·9 cm. (Begin with circumcircle.)

19. Construct an isosceles triangle, $a=b=1{\cdot}7''$, $\mathsf{R}=1{\cdot}1''$.

20. Construct a right triangle, $a=1{\cdot}8$ cm., $\mathsf{R}=2{\cdot}3$ cm.

21. Construct a square on a side of 3·3 cm.

22. Draw any straight line **AB**, find the locus of a point **P** which is always 1″ from **AB**.

23. Find a point equidistant from a given point and a given line. (Take 2·2 cm. from each.)

24. Draw an angle of 58°. Construct a point 2 cm. from each of its sides.

25. Construct the locus of points equidistant from the sides of an angle of 46°.

26. Construct a right triangle, hyp. $a=2''$, $b+c=2\frac{5}{8}''$.

27. Construct a triangle, $a=3{\cdot}5$ cm., $b+c=5{\cdot}3$ cm., $\mathsf{A}=78°$.

28. Construct a triangle, $a+b+c=3''$, $\mathsf{B}=38°$, $\mathsf{C}=72°$.

29. Construct a parallelogram, sides $1\frac{5}{8}''$, $2\frac{1}{2}''$, one diag. $3\frac{1}{4}''$.

30. Draw a straight line **AB**, take two points **C, D** on opposite sides of **AB**, construct a point **P** in **AB** so that **AB** bisects angle **CPD**.

31. Construct a circle, radius $1{\cdot}3''$, to pass through two points **A, B**, $2''$ apart. Can it be done if $\mathsf{AB}=3''$?

32. Construct a point equidistant from two parallel lines $1\frac{1}{2}''$ apart.

33. Construct the locus of a point equidistant from two parallels.

34. Draw two parallels, and take two points, one between the parallels and one outside them; construct a point equidistant from the parallels, and also equidistant from the two points.

35. Construct a point in one side of an angle whose (perp.) distance from the other is a given length.

36. Through a given point **P** draw a straight line to make equal angles with the two sides of a given angle. How many such lines can be drawn?

37. Through a given point **A** draw a straight line making a given angle with a given straight line **BC**.

38. Through two given points **P, Q** draw straight lines to form with a given line **BC** an equilateral triangle **ABC**. (Use Ex. 37, and angles of 60°.)

39. Draw two straight lines at an angle of 108°; construct a point $1''$ from one line, $1\frac{1}{8}''$ from the other.

40. From a ship at sea, a second ship is 3 miles E., and a third is 5 miles N.W. Find by a diagram the distances between the last two.

41. Could you do Ex. 40 in the same way if the distances were 3000 miles E. and 5000 miles N.W.? Why?

42. Draw a circle, $1\frac{1}{4}''$ radius. Draw a chord in it whose arc has an angle of 76°.

43. Draw a triangle, $a=4{\cdot}7$ cm., $\mathsf{A}=67°$, altitude from **BC**, 3·2 cm.

44. A ship is 5 miles from shore, and the directions from the ship of two lighthouses on the shore, 8 miles apart, form an angle of 70°. Draw a plan, and measure the distances of the ship from the lighthouses.

45. A boatman heading for shore sees a tower, close to the shore, 45° to his right; after making two miles he sees it 60° to his right. What was his distance from shore?

46. Find the locus of mid points of all chords of a circle through a given point P on the circle.

47. Construct a square, diagonal 2·83″.

48. Construct a rhombus, diagonal 6 cm., opp. ang. 98°.

49. Construct a rhombus, diagonals 3·7 cm., 2·9 cm.

50. The sides of a jointed parallelogram ABCD are AB, 2·5 cm., BC, 3 cm. If BC is fixed, find the locus of the points A, D as the parallelogram moves.

51. Construct a rectangle, sides 2·7″ and 1·9″, and draw the right bisector of the first side. Show by symmetry that it is the right bisector of the opposite side.

52. Draw two diameters at an angle of 60°, in a circle of 1″ radius. Show that their ends form a rectangle. Measure the sides.

53. Find the greatest line that can be drawn in a parallelogram from a vertex to one of the sides. Give proof.

54. Construct a triangle, given $a=4$ cm., $b-c=1$ cm., C=54°.

55. Find the greatest chord of a circle. Give proof.

56. Construct a trapezium, AB ∥ CD, ang. A=B=60°, AB=5 cm., AD=3 cm. Measure BC, BD.

57. One angle of an isosceles triangle is half each of the others. Calculate the angles.

58. On a base of 3·4 cm. describe an isosceles triangle, having base angles each double of the third angle. (See Ex. 57.) Measure the equal sides.

59. On a base BC of 1″ draw an isosceles triangle having ang. B=C=2A. Draw CD to AB, bisecting angle C. Measure CD, AD. What kind of triangle is CDA? Calculate its angles.

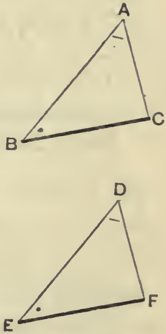
60. Construct the triangle ABC and the bisector CD of Ex. 59. On AB, AC outwards, describe isosceles triangles AEB, AFC, equal sides equal to BC. What is the figure ADCF? Calculate the angles A, F, C of the pentagon AEBCF. What kind of pentagon is it?

CHAPTER III.

CONGRUENT TRIANGLES—SIMILAR FIGURES—AREAS.

(The symbol $\equiv$ is used for 'is congruent to.')

Theorem 25.—'Two triangles are congruent which have one side and two angles of one equal respectively to one side and two corresponding* angles of the other.'



If the triangles ABC, DEF have any two angles A = D, B = E, then also third angle C = F.

If also one side BC = EF, place tr. DEF on ABC, reversing if necessary, so that EF coincides with BC, ang. E with B, and ang. F with C;

$\therefore$ sides ED, FD fall on BA, CA;

$\therefore$ point D coincides with A;

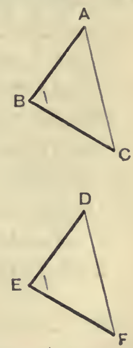
$\therefore$ triangle DEF $\equiv$ ABC.

Note. The corresponding sides are equal—

viz. AB = DE, opp. angles C, F;

AC = DF, 〃 B, E.

Theorem 26.—'Two triangles are congruent which have two sides and their angle of one equal respectively to two sides and their angle of the other.'



If the triangles ABC, DEF have AB = DE, BC = EF, and their ang. B = ang. E; place tr. DEF on ABC, so that ang. E coincides with B, and side EF with BC;

$\therefore$ D coincides with A, $\because$ ED = BA;

$\therefore$ triangle DEF $\equiv$ ABC.

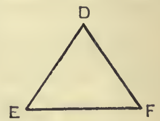
Ex. State what sides and angles are equal in consequence.

* In congruent triangles those sides correspond which are opposite to equal angles; and angles correspond which are opposite to equal sides.

Theorem 27.—'Two triangles are congruent which have three sides of one equal respectively to three sides of the other.'

If the triangles ABC, DEF have
AB = DE, BC = EF, CA = FD;
draw arc AG, centre B,
and arc AH, centre C.

These circles meet again, once only, on the *other* side of the line of centres BC.

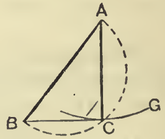


Place tr. DEF on ABC, on the *same* side of BC, so that EF coincides with BC;
then D falls on arc AG, ∵ ED = BA;
and D also falls on arc AH, ∵ FD = CA;
∴ D coincides with A;
∴ triangle DEF ≡ ABC.

Ex. (i.). State what angles are equal.

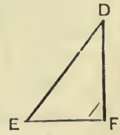
Ex. (ii.). A diagonal divides a rhombus into congruent triangles.

Theorem 28.—'Two right triangles are congruent which have the hypotenuse and one side of one equal respectively to the hypotenuse and one side of the other.'



If the right triangles ABC, DEF have
hyp. AB = DE, side AC = DF;
draw semicircle ACB, diam. AB,
and arc CG, centre A.

These circles meet again, once only, on the *other* side of the line of centres AB.

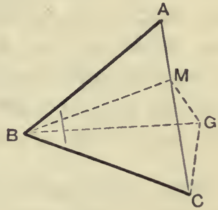


Place tr. DEF on ABC on the *same* side of AB, so that DE coincides with AB;
then F falls on semcle. ACB, ∵ ang. F is right;
and F falls also on arc CG, ∵ DF = AC;
∴ F coincides with C;
∴ triangle DEF ≡ ABC.

Ex. (i.). State what sides and angles are equal.

Ex. (ii.). If a quadrilateral ABCD has A, C right angles, and side AB = CB, the triangles ABD, CBD are congruent.

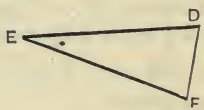
Theorem 29.—'If a triangle has two sides equal respectively to two sides of another, but their angle greater than that of the equal sides of the other, its third side is also greater than that of the other.'



If the triangles ABC, DEF have sides AB = DE, BC = EF, but their ang. B > E; place tr. DEF on ABC in position GBC, so that EF coincides with BC.

Draw BM bisecting ang. ABG, join MG.

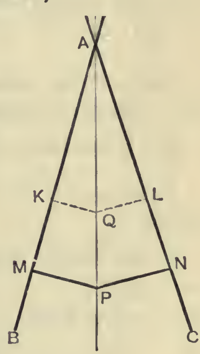
Then by symmetry about BM,
AM = MG, ∵ BA = BG;
∴ AC, i.e. AM + MC = MG + MC
> GC;
i.e. AC > DF.



Ex. Prove the converse of this theorem.

Theorem 30.—'The locus of a point in a plane equidistant from two lines in the plane is the bisectors of their angles.'

If P is a point equidistant from the lines AB, AC, perp. PM = PN;*
then in rt. trs. PMA, PNA,
PM = PN, hyp. PA is common,
∴ ang. PAM = PAN;
i.e. any point on the locus is on a bisector of angle of the lines.

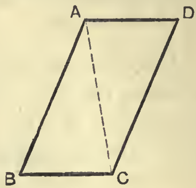


Also, if Q is on a bisector of angle;
then in rt. trs. QKA, QLA,
ang. K = L, ang. QAK = QAL,
QA is common;
∴ perp. QK = QL;
i.e. any point on a bisector of angle is on the locus;
∴ the locus is the bisectors of angles of AB, AC.

Ex. State the theorem when the two lines are parallel.

* The distance from a point to a straight line generally means the shortest—i.e. perpendicular—distance.

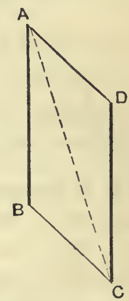
Theorem 31.—'Opposite sides and angles of a parallelogram are equal; and a diagonal divides it into congruent triangles.'



If AC is a diag. of parm. ABCD;
then in trs. ABC, CDA, AC is common,
ang. BAC = alt. ang. DCA,
ang. BCA = alt. ang. DAC;
∴ triangle ABC ≡ CDA,
and side AB = opp. side DC, BC = AD;
also ang. B = D; similarly, A = C.

Cor.—'The perpendicular distance between two parallels is everywhere the same.'

Theorem 32.—'The joins towards the same parts of the ends of two equal and parallel straight lines are equal and parallel.'

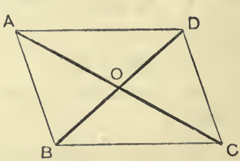


If AB = and || CD, and AD, BC are joins towards same parts;
then in trs. ABC, CDA,
BA = DC, AC is common,
ang. BAC = alt. ang. DCA;
∴ BC = AD; i.e. the joins are equal;
and ang. ACB = CAD, the alt. ang.;
∴ BC || CD; i.e. the joins are parallel.

Cor.—'The locus of a point at a given distance from a straight line is two parallels at that distance.'

Ex. What kind of figure is ABCD?

Theorem 33.—'The diagonals of a parallelogram bisect each other.'

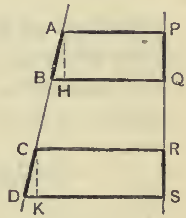


If the diags. AC, BD of parm. ABCD meet in O;
then in trs. ABO, CDO,
AB = CD, ang. OAB = alt. ang. OCD,
ang. OBA = alt. ang. ODC;
∴ OB = OD, and OC = OA;
i.e. AC and BD bisect each other at O.

Ex. If two straight lines bisect each other, the joins of their ends form a parallelogram.

Theorem 34.—'A system of parallels which cuts off equal parts from one straight line cuts off equal parts from any other which it meets.'

If a system of parls. cuts off equal parts AB, CD from one line AD,
and cuts off parts PQ, RS from another PS;
draw AH, CK parl. to PS.

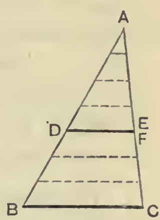


Then in trs. ABH, CDK,
AB = CD, ang. A = C (same parts),
ang. B = D;
∴ AH = CK.

Also PQ = AH in parm. AQ,
and RS = CK in parm. CS,
∴ PQ = RS.

Theorem 35.—'A parallel to one side of a triangle cuts off the same fractional part from the other two sides; and a straight line which cuts off the same fractional part from two sides of a triangle is parallel to the third.'

If ABC is a triangle, and DE parl. to BC cuts off $\frac{4}{7}$, say, of AB;
divide AB into 7 equal parts so that AD contains 4 of them;
then a system of parls. through the points of division divides AC into 7 equal parts, of which AE contains 4;
i.e. DE cuts off $\frac{4}{7}$ of AB and AC.



(ii.) If DF cuts off the same fractional part, $\frac{4}{7}$ say, from the sides AB, AC of the triangle;
make DE parl. to BC;
∴ AE = $\frac{4}{7}$AC = AF.
∴ E coincides with F;
i.e. DF ∥ BC.

Prove similarly for any other fraction.

Ex. Show that DE = $\frac{4}{7}$BC.

NOTE ON MEASURE, RATIO, AND PROPORTION.

This may be taken, as required, with the theorems on proportion in this chapter.

MEASUREMENT.

Definition of Measure.—The **measure** of any magnitude is the number expressing how much of the unit magnitude it contains.

This **number** is defined as **greater or less** according as the magnitude it measures is greater or less.

The measures of magnitudes can sometimes be expressed (as $PQ = 2\frac{1}{3}''$, above) as arithmetical fractions, and are then *rational;* but just as often measures cannot be so expressed (as RS, the diag. of a square, unit side, above), and are then *irrational.*

Each kind of measure can, however, be expressed by means of an infinitely continued decimal, thus:

Make a scale, and measure PQ or RS by setting P or R on a unit division, and dividing the unit containing Q or S decimally into tenths, hundredths, and so on indefinitely.

Then Q will be found between the 3rd and 4th tenths, 3rd and 4th hundredths, and so on.

Thus, measure of PQ is 2·333...*ad inf.*

Similarly, measure of RS is 1·414213 *...*ad inf.*

If PQ′ is any length less than PQ, some of the scale divisions will come between Q and Q′; hence:

(i.) '**Unequal magnitudes are measured by different decimals;**
(ii.) **Magnitudes measured by the same decimal are equal;**
(iii.) **Each decimal represents one only number.'**

These decimals can be multiplied, &c., as in ordinary arithmetic, working from the *left.*†

* Irrational numbers do not give recurring decimals.

† This is proved in note on ratio at the end of Chapter V.

(Capital letters denote magnitudes, Greek letters numbers.)

Definition of Ratio.—The **ratio** of a magnitude to another of the same kind * is the number expressing how much of the second is contained in the first.

The first is the **antecedent** or **numerator**, the second the **consequent** or **denominator**, of the ratio.

The ratio of X to Y is written $X:Y$, or $\frac{X}{Y}$; and if μ is the number we write, $X:Y=\mu$, $\frac{X}{Y}=\mu$, or $X=\mu Y$.

Construction of a ratio $X:Y=\mu$:

Find the measure μ of X on a decimal scale, unit Y;

$\therefore$ $X:Y=\mu$.

Thus the ratio PQ : 1 inch, on the last page, is $2{\cdot}\dot{3}$; and diagonal of square : side of square is $1{\cdot}414213\ldots(=\sqrt{2})$.

Proportion consists in the equality of ratios.

Four magnitudes are in proportion when the ratio of the 1st to the 2nd is equal to that of the 3rd to the 4th;

i.e. X, Y, Z, W are in proportion when $X:Y=Z:W$.

X, W are the **extremes**, Y, Z the **means** of the proportion; W is the **fourth proportional** of X, Y, Z.

Three magnitudes are in proportion when the ratio of the 1st to the 2nd is equal to that of the 2nd to the 3rd;

i.e. X, Y, Z are in proportion when $X:Y=Y:Z$.

Y is the **mean proportional** of X, Z.

Z is the **third proportional** of X, Y.

The **mean part** of a magnitude is the mean proportional of the whole magnitude and the other part.

A straight line is divided **harmonically** in four points A, B, C, D when two points B, D divide the line AC between the others, internally and externally in the same ratio.

Note. The numerical definition of ratio is required only for comparing magnitudes of one kind with those of another kind. For geometry of figure, ratio may be treated entirely geometrically. (See Ch. VII.)

* Two magnitudes are of the same kind when one is greater than, equal to, or less than the other.

If Z is any unit, and $X = \lambda Z$, $Y = \nu Z$, and ratio $X : Y = \mu$; then $\lambda Z = X = \mu Y = \mu\nu Z$;
$\therefore\ \lambda = \mu\nu$. (Def. of equal number.)

Hence μ is the ratio $\lambda : \nu$; i.e. the fraction* or quotient $\frac{\lambda}{\nu}$ expressing how much of the number ν is contained in λ;
i.e. the ratio $X : Y =$ the fraction λ/ν.

Hence ratios $X : Y$, &c., may be treated by the ordinary rules of fractions in algebra, when the results are intelligible.

Unit Theorem.—'Two magnitudes are equal (i.) which have the same ratio to a third; (ii.) to which a third has the same ratio.'

$X = Y$ (i.) if $\frac{X}{Z} = \frac{Y}{Z}$, or (ii.) if $\frac{Z}{X} = \frac{Z}{Y}$.

Product Theorem.—'If four magnitudes are proportional, the product of extremes is equal to the product of means.'

If $\frac{X}{Y} = \frac{Z}{W}$, then $XW = YZ$.

Note. If X, Y, Z, W are lengths, their products represent rectangles. Thus, 'If four straight lines are proportional, the rectangle of extremes is equal to the rectangle of means.'†

Alternando.—'If four magnitudes of the same kind are proportional, the second and third terms can be interchanged.'

If $\frac{X}{Y} = \frac{Z}{W}$, then $\frac{X}{Z} = \frac{Y}{W}$.

Summation.—'If $\frac{X_1}{Y_1}$, $\frac{X_2}{Y_2}$, $\frac{X_3}{Y_3}$, &c., are equal ratios of one kind of magnitude, each ratio $\frac{X_1}{Y_1} = \frac{X_1 + X_2 + X_3 + \&c.}{Y_1 + Y_2 + Y_3 + \&c.} = \frac{X_1 - X_2}{Y_1 - Y_2}$, and so on.'

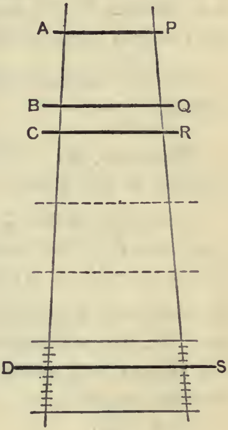
Other theorems may be assumed from algebra as required.

These theorems are formally proved in Ch. V., note.

* Since λ and ν are numbers.

† A strict geometrical proof of this is given in Th. 52, below.

Theorem 36.*—'A system of parallels cuts off proportional parts from any two straight lines which it meets.'



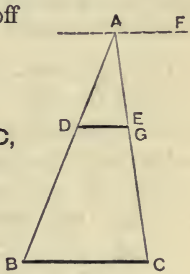
If a system of parls. cuts off AB, CD, and PQ, RS from the two lines;
make a scale, unit AB, from C along CD, divide decimally the unit containing D into tenths, hundredths, &c., draw parls. to AP, &c., through the points of division;
these form a similar scale from R, unit PQ.

Also, since DS || system of parls. AP, &c., the points D and S come between the same divisions respectively of the scales of AB, PQ;

∴ ratio $\frac{CD}{AB}=\frac{RS}{PQ}$, since both are represented by the same decimal.

Ex. Name all the sets of proportional parts.

Theorem 37.*—'A parallel to one side of a triangle cuts off proportional parts from the other two; and a straight line which cuts off proportional parts from two sides of a triangle is parallel to the third.'



If ABC is a triangle, and DE || BC;
make AF parl. to BC;
then the system of parls. AF, DE, BC cuts off propl. parts from AB, AC;

i.e. $\frac{AD}{AB}=\frac{AE}{AC}$, &c.

(ii.) If DG cuts off propl. parts from AB, AC,

so that $\frac{AG}{AC}=\frac{AD}{AB}$, make DE parl. to BC;

$\therefore \frac{AG}{AC}=\frac{AD}{AB}=\frac{AE}{AC}$;

∴ AG = AE, and G coincides with E;
i.e. DG || BC.

Ex. Name all sets of proportional parts.

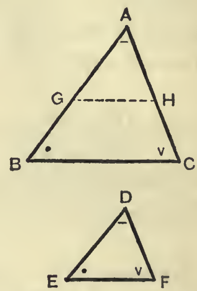
* These may be postponed by beginners. They are, however, general, and apply to rational and irrational numbers alike.

(The symbol ||| is used for 'is similar to.')

Definition 26.—Two **polygons** of the same number of sides are **similar** whose angles in the same order are respectively equal, and whose corresponding sides are proportional.

Cor.—'**Regular polygons of the same number of sides are similar.**'

Theorem 38.—'Two triangles are similar which have two angles of one equal respectively to two angles of the other.'



If the trs. ABC, DEF have
ang. A = D, B = E, and ∴ C = F;
place tr. DEF on ABC in position AGH,
so that ang. D coincides with A,
and DE, DF with AG, AH;
∴ ang. G = B, and GH || BC;

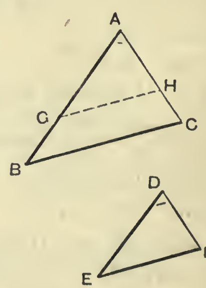
$$\therefore \frac{AG}{AB} = \frac{AH}{AC};$$

$$\text{i.e. } \frac{DE}{AB} = \frac{DF}{AC} = \text{(similarly) } \frac{EF}{BC};$$

∴ tr. DEF ||| ABC.

Note. The sides opp. the eql. angs. are propl. and are called corresponding sides.

Theorem 39.—'Two triangles are similar which have two sides of one proportional to two of the other, and the angles of these sides equal.'



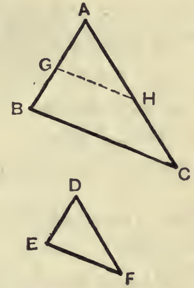
If the trs. ABC, DEF have $\frac{DE}{AB} = \frac{DF}{AC}$,
and ang. D = A;
place tr. DEF on ABC in position AGH,
so that ang. D coincides with A,
and DE, DF with AG, AH;

$$\text{then } \frac{AG}{AB} = \frac{DE}{AB} = \frac{DF}{AC} = \frac{AH}{AC};$$

∴ GH || BC;
∴ ang. B = G (same parts) = E, and C = F;
∴ tr. DEF ||| ABC.

Ex. If two triangles have corresponding sides parallel, show that they are similar.

Theorem 40.—'Two triangles are similar which have three sides of one proportional respectively to three sides of the other.'



If the trs. ABC, DEF have

$\frac{DE}{AB}=\frac{EF}{BC}=\frac{DF}{AC}$;

make AG eql. to DE, and GH parl. to BC;

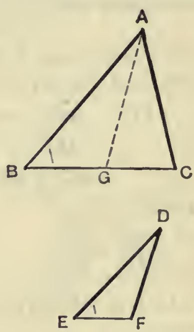
$\because \frac{AH}{AC}=\frac{AG}{AB}=\frac{DE}{AB}=\frac{DF}{AC}$;

$\therefore$ AH = DF; similarly, GH = EF;

also AG = DE in trs. AGH, DEF;

$\therefore$ tr. DEF $\equiv$ AGH $|||$ ABC.

Theorem 41.—'If two triangles have one angle of one equal to one of the other, and the sides of a second angle of each proportional; then either (i.) the third angles are equal, and the triangles similar, or (ii.) the third angles are supplements.'



If trs. ABC, DEF have ang. B = E, and sides of angs. A, D proportional,

viz. $\frac{AB}{DE}=\frac{AC}{DF}$; then

either (i.) ang. A = D; but B = E,

$\therefore$ C = F, and tr. DEF $|||$ ABC;

or (ii.) ang. A $\ne$ D;

make BAG eql. to D;

$\therefore$ BGA = F ($\because$ B = E);

$\therefore$ tr. DEF $|||$ ABG;

$\because \frac{AG}{DF}=\frac{AB}{DE}=\frac{AC}{DF}$;

$\therefore$ AG = AC;

$\therefore$ ang. C = AGC = suppt. of AGB

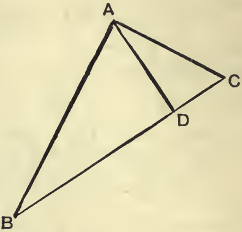
= suppt. of F.

Cor.—'Two right triangles are similar which have the sides of one acute angle of each proportional.'

(Make B, E rt. angs.)

Ex. Write the theorem when AB, AC = DE, DF respectively.

Theorem 42.—'**A right triangle is divided into similar triangles by the perpendicular from the right angle.**'



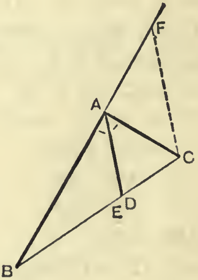
If A is the right angle in rt. tr. ABC,
and AD the perp. from A;
then ang. DAC = compt. of BAD = B,
and rt. ang. D = A;
∴ tr. DAC ||| ABC.

Similarly, tr. DBA ||| ABC ||| DAC.

Note. Corresponding sides are opposite equal angles;
e.g. $\frac{CD}{AC}$ (short sides) $= \frac{DA}{AB}$ (long sides) $= \frac{AC}{BC}$ (hyp.).

Ex. Show that BA is the mean propl. of BD, BC.

Theorem 43.—'**The bisector of an angle or exterior angle of a triangle divides the opposite side in the ratio of the other sides; also (ii.), a straight line through an angle of a triangle dividing the opposite side in the ratio of the other two is a bisector of the angle.**'



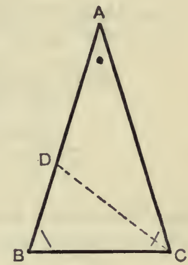
If AD bisects ang. A of tr. ABC,
make CF parl. to AD;
∴ ang. AFC = BAD, same parts,
= DAC
= ACF, alt. ang.;
∴ AC = AF,
and $\frac{BD}{DC} = \frac{BA}{AF} = \frac{BA}{AC}$.

Also (ii.), if AE divides BC so that
$\frac{BE}{EC} = \frac{BA}{AC}$,
make CF parl. to bisr. AD;
then $\frac{BE}{EC} = \frac{BA}{AF}$, ∵ AF = AC;
∴ AE || CF, and coincides with bisector AD.

Ex. Prove the theorem for the exterior ang. at A; and show that the two bisectors divide BC harmonically.

Theorem 44.—'An isosceles triangle whose unequal side is the mean part * of the equal sides has the equal angles each double the third angle.'

If isosceles tr. ABC has BC the mean part of eql. sides AB, AC; make AD eql. to BC, join CD;



∴ AD is the mean part of AB;

$\therefore \frac{BD}{DA} = \frac{DA}{BA} = \frac{BC}{CA}$;

∴ DC bisects ang. BCA in tr. ACB;

also, ang. B is common to trs. DBC, CBA,

and ratio of sides $\frac{DB}{BC} = \frac{DB}{DA} = \frac{BC}{BA}$;

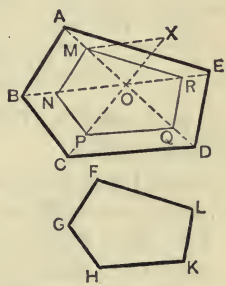
∴ tr. DBC ||| BCA;

∴ ang. $A = DCB = \frac{1}{2}BCA = \frac{C}{2} = \frac{B}{2}$.

Definition 27.—Polygons are **similarly situated** whose corresponding sides are parallel, and whose joins of corresponding vertices pass all through a fixed point.

Theorem 45.—'Two similar polygons can be placed so as to be similarly situated about any chosen point.'

If FGHKL ||| ABCDE, in order of letters; take any point O, join OA, OB, &c.; make OX parl. to AB, eql. to FG, XM parl. to OB; make MN, NP, PQ, &c., parl. to AB, BC, CD, &c.



Then ang. MNP = ang. B = G; and MN = OX, in parm. XN, = FG;

also $\frac{NP}{BC} = \frac{ON}{OB} = \frac{MN}{AB} = \frac{FG}{AB} = \frac{GH}{BC}$;

∴ NP = GH.

Similarly, ang. NPQ = C = H, and PQ = HK, and so on; ∴ fig. FGHKL ≡ MNPQR, and may be placed in position MNPQR, similarly situated to ABCDE about O.

Ex. If corresponding sides of similar polygons are parallel, or joins of corresponding points concurrent, the polygons are similarly situated.

* See Def., p. 65, near bottom of page.

Theorem 46.—'The perimeters of two similar polygons are proportional to corresponding sides.'

If ABCDE, FGHKL are similar figures, in the order of the letters (see last fig.),

$$\frac{AB}{FG}=\frac{BC}{GH}=\frac{CD}{HK}=\frac{DE}{KL}=\frac{EA}{LF}$$
$$=\frac{AB+BC+CD+DE+EA}{FG+GH+HK+KL+LF}\text{ (summation)}$$
$$=\frac{\text{perimeter of ABCDE}}{\text{perimeter of FGHKL}}.$$

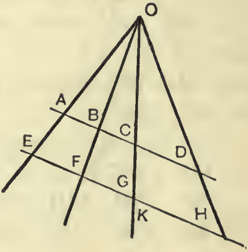
Note. This is true of similar triangles. Deduce a construction for a triangle, angles and sum of sides given.

Theorem 47.—'Concurrent straight lines divide similarly any two parallels which they meet, and (ii.) the joins of corresponding points of similarly divided parallels are concurrent.'

(i.) If concurrent lines through O cut the parallels in A, B, C, D and E, F, G, H respectively;

then $\frac{AB}{EF}=\frac{OB}{OF}=\frac{BC}{FG}$

$=(\text{similarly})\ \frac{CD}{GH}.$



(ii.) If ABCD, EFGH are similarly divided parallels;
draw EA, FB to meet in O,
and OC to meet EH in K;
$\therefore \frac{BC}{FK}=\frac{AB}{EF}=\frac{BC}{FG};$
$\therefore$ FK = FG, and G coincides with K;
$\therefore$ GC passes through O;
similarly, DH passes through O;
i.e. the joins EA, FB, GC, HD are concurrent.

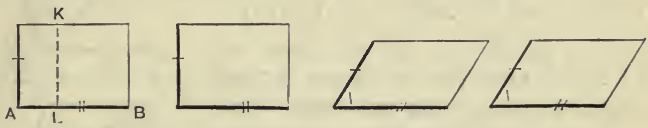
Note. The proof is general, and applies to any number of points satisfying the given conditions. Also, the lines are similarly situated, and this is a particular case of Th. 45.

(The symbols $\triangle$, ▱ are used for area of triangle, parallelogram.)

Theorem 48.—(i.) 'Two rectangles or two parallelograms of given angle, which have the sides of one equal respectively to those of the other, are congruent and have equal areas.'

(ii.) 'Two rectangles of equal areas and bases have equal altitudes.'

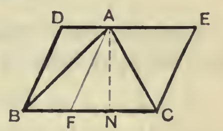
(iii.) 'Two squares of equal areas have equal sides.'



One figure can be placed to coincide with the other.

Note. We may denote a rectangle by its base and altitude—thus, rect. AB . KL—since these completely determine it.

Theorem 49.—'A triangle has half the area of a rectangle or parallelogram of the same base and altitude.'



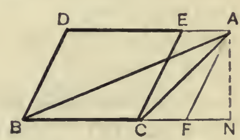
If tr. ABC and rect. or parm. DBCE have same base BC, and same altitude AN; then A lies in DE parl. to BC.

Make AF parl. to BD and CE;

then $\triangle$ ABF $= \frac{1}{2}$ ▱ DF,

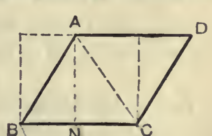
and $\triangle$ ACF $= \frac{1}{2}$ ▱ EF;

$\therefore$ sum or diffce. $\triangle$ ABC $= \frac{1}{2}$ ▱ DC.



Cor.—$\triangle$ ABC $= \frac{1}{2}$ rect. BC . AN.

Theorem 50.—'A parallelogram has the area of a rectangle of the same base and altitude.'



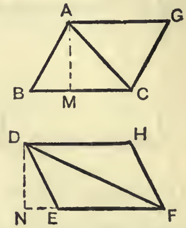
If ABCD is a parm., base BC, alt. AN; ▱ AC $= 2$ $\triangle$ ABC $=$ rect. BC . AN of same base and altitude.

Ex. A trapezium has the area of the rectangle of the altitude of its parallel sides and half their sum. (Divide into triangles.)

Theorem 51.—(i.) 'Two triangles or two parallelograms of equal bases and altitudes have equal areas.'

(ii.) 'Two triangles or two parallelograms of equal areas and bases have equal altitudes.'

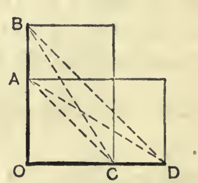
(i.) If trs. ABC, DEF or parms. AC, DF have base BC = EF, and alt. AM = DN;
then △ ABC = $\frac{1}{2}$ rect. BC . AM = $\frac{1}{2}$ rect. EF . DN
= △ DEF;
and ▱ AC = rect. BC . AM = rect. EF . DN
= ▱ DF.



(ii.) If trs. ABC, DEF or parms. AC, DF have equal areas, and base BC = EF;
then rect. BC . AM = 2 △ ABC = 2 △ DEF
= rect. EF . DN;
∴ alt. AM = DN;
and rect. BC . AM = ▱ AC = ▱ DF
= rect. EF . DN;
∴ AM = DN.

Theorem 52.—'The rectangles of the extremes and means of four straight lines in proportion are equal.'

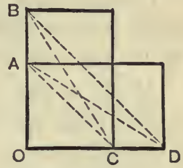
If a, b, c, d are four straight lines in propn., $a : b = c : d$;
make OA eql. to a, OB eql. to b,
along one line,
OC eql. to c, OD eql. to d,
along a perp.;
complete rects. AD, BC, join AC, &c.
Then AC ∥ BD, ∵ OA : OB = OC : OD;
∴ △ ACD = △ ACB, same base and alt.;
∴ whole △ OAD = △ OBC;
∴ rect. AD = rect. BC;
i.e. rect. $a . d$, of extremes = rect. $b . c$, of means.



Cor.—'If three straight lines are proportional, the rectangle of extremes is equal to the square of the mean.'

Ex. Prove the theorem for two parallelograms or two triangles of given angle.

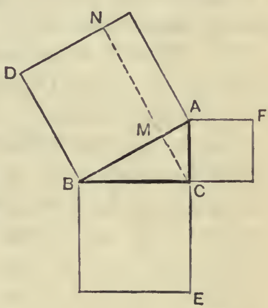
Theorem 53.—'Two equal rectangles have their sides the extremes and means of a proportion.'



Place the equal rects. AD, BC to have an angle common at O;
then △ OAD = △ OBC; take away OAC;
∴ △ ACD = △ ACB, of same base AC;
∴ trs. ACD, ACB have same alt.;
∴ BD ∥ AC, and OA : OB = OC : OD;
i.e. OA, OD are extremes, and OB, OC means, of a proportion.

Theorem 54.—(i.) 'The square on the hypotenuse of a right triangle is equal to the sum of squares on the other sides.' *

(ii.) 'A triangle is right which has the square on one side equal to the sum of squares on the other sides.'



(i.) If AD, BE, CF are squares on the hyp. and sides of rt. tr. ABC;
draw CMN perp. to AB, parl. to BD.

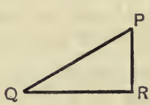
Then tr. CBM ||| ABC;

∴ $\frac{BM}{BC} = \frac{BC}{BA}$;

∴ sq. on BC = rect. BM . BA = rect. BN.

Similarly, sq. on AC = rect. AM . AB
= rect. AN;

∴ sq. on BC + sq. on AC
= rect. BN + rect. AN = fig. AD
= sq. on AB.



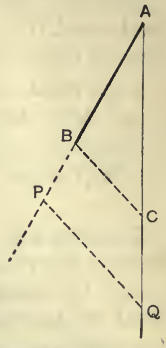
(ii.) If triangle PQR has
sq. on PQ = sq. on QR + sq. on PR;
make BCA a right angle,
BC = QR, AC = PR;
∴ sq. on PQ = sq. on QR + sq. on PR = sq. on BC + sq. on AC
= sq. on AB;
∴ PQ = AB; and in trs. PQR, ABC,
PQ = AB, QR = BC, PR = AC;
∴ ang. R = C = a right angle.

Ex. The sum of squares on the sides of a rectangle is equal to the sum of squares on the diagonals.

* For proof by dissection, see p. 36, Ch. I.

PROPORTIONALS.

Construction 11.—'Construct a straight line having a given ratio μ to a given line' (i.e. given a line X and number μ, construct μX).*



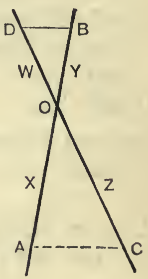
If AB is the line, set off a unit AC at an angle to AB, calculate μ as a decimal $a{\cdot}a_1a_2\ldots$; make $AQ = a{\cdot}a_1a_2\ldots$ units, to as many decimal places as possible or required; make QP parl. to CB;

$\therefore$ AP : AB = AQ : AC = μ;

i.e. AP has the ratio μ to AB.

Note. If the ratio is given as that of two lines l, m, AP can be found by the following construction.

Construction 12.—'Construct the fourth proportional W of three straight lines X, Y, Z.'



From a point O set off OA, OB equal to X, Y along one line,† and OC eql. to Z along another; make BD parl. to AC.

Then X : Y = OA : OB = OC : OD = Z : OD;

$\therefore$ OD is the 4th propl. W of X, Y, Z.

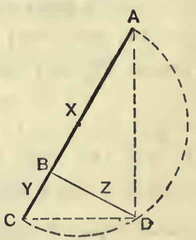
Similarly, if X, Z, W are given, Y is constructed by making DB parl. to AC.

Thus any term in a proportion of lines can be constructed when three are given.

Ex. 1. Construct the third proportional Z of X, Y.

Ex. 2. Given 1st, 2nd, and 4th terms X, Y, W, construct the 3rd Z.

Construction 13.—'Construct the mean proportional Z of two lines X, Y.'



Make AB, BC eql. to X, Y, in one line; draw semicircle ADC, and BD perp. to AC.

Then rt. tr. ABD ||| DBC;

$\therefore$ AB : BD = BD : BC;

i.e. BD is the mean propl. Z of X, Y.

Note. Make X a unit, Y n units;

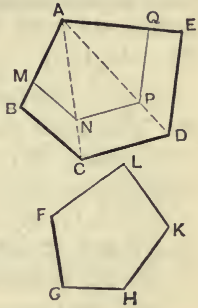
$\therefore$ Z = $\sqrt{n}$ units. We thus construct $\sqrt{n}$.

Construct thus $\sqrt{3}$, $\sqrt{5}$, and measure.

* The theoretical construction is given at the end of Ch. V.

† A and B need not be on opp. sides of O.

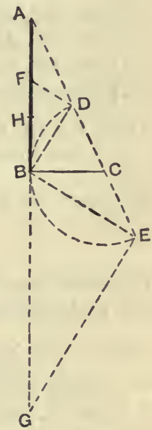
Construction 14.—'Construct a polygon on a given side, similar to a given polygon.'



If ABCDE is the polygon, FG the side corresponding to AB,
join AC, AD;
make AM = FG; MN, NP, PQ parl. to sides of polygon;
on FG make tr. FGH ≡ AMN;
" FH " FHK ≡ ANP,
and so on;
∴ fig. FGHKL ≡ AMNPQ ||| ABCDE.

Construction 15.—'Construct the line of which a given line is the mean part;* and divide a line in mean section† (i.e. so that one part is the mean part of the line).'

If AB is the line, make BC half AB and perp. to AB;
draw semicircle DBE, centre C, to cut AC in D, E, so that DE = AB; draw DF parl. to BE.



Then tr. ADB ||| ABE, ∵ ang. A is common, and ang. ABD = compt. of CBD
= CBE (semicircle) = AEB (∵ CE = CB).

Also DE = AB;

$$\therefore \frac{AD}{DE} = \frac{AD}{AB} = \frac{AB}{AE} = \frac{DE}{AE};$$

i.e. AE is the line whose mean part is DE or AB;
and AFB is divided similarly to ADE;
∴ BF is the mean part of AB,
and AB is divided in mean at F.

Note. (i.) AD is also the mean part of AB; hence AD = BF; thus we may construct by drawing arc DH, centre A, to meet AB in H.

(ii.) If EG ∥ DB, AG is divided in mean at B; i.e. AB is divided externally so that one part BG is the mean between the line AB and the other part AG.

Ex. Divide a straight line so that the square on one part is equal to the rectangle of the whole and the other part.

* See definition at foot of p. 65.

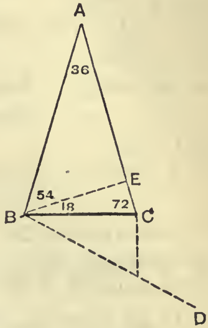
† Also called 'in medial section' and 'in extreme and mean ratio.'

Construction 16.—**'Construct an isosceles triangle whose base is the mean part of the equal sides, and whose equal angles are each double the third angle.'** (Angles of 72° and 36°, 54°, 18°.)

If BC is the base, construct BD of which BC is the mean part;
with centres B, C, radius BD, draw arcs A;
ABC is the triangle.

By Th. 44, ang. $B = C = 2A$.
Also $A + B + C = 5A = 180°$;
$\therefore$ $A = 36°$, $B = 72°$.

Draw BE perp. to AC;
$\therefore$ ang. EBC = compt. of $C = 18°$,
and $EBA = 54°$.



Note. We can construct by ruler and compass only,

(i.) 90°, 45°, by right angle and bisection.
(ii.) 60°, 30°, 15°, by equil. triangle and bisection.
(iii.) 72°, 54°, 36°, 27°, 18°, 9°, as above and by bisection.
(iv.) Any angle obtained from these, as sum or difference.

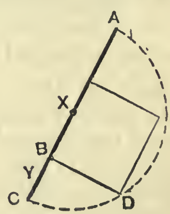
These are all the angles in whole numbers of degrees that can be drawn in this manner.

Ex. Show that the complete series 3°, 6°, 9°...(adding 3° each time) can be obtained.

Construction 17.—**'Construct a square equal in area to a given rectangle of sides X, Y.'** *

Make $AB = X$, $BC = Y$, in one line;
draw semcle. ADC, make BD perp. to AC,
and draw sq. on BD.

Then BD is mean propl. of X, Y;
i.e. $X : BD = BD : Y$;
$\therefore$ sq. on BD = rect. $X . Y$.



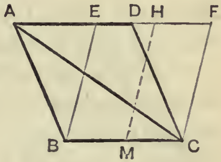
Note. The next constructions show how to make a rectangle equal to a given triangle or polygon; thus a square can be constructed equal in area to any polygon.

* Reduces to construction of mean propl.

Construction 18.—'Draw a rectangle, or parallelogram of given angle, equal in area to a given parallelogram or triangle.'

If ABCD is a given parm., make CBE the given ang., draw parm. EBCF;
∴ ▱ EC = ▱ AC, same base and alt.

If ABC is a given tr., make BM half BC, ang. CBE the given ang., draw parm. EM;
∴ ▱ EM = 2 △ ABM = △ ABC.



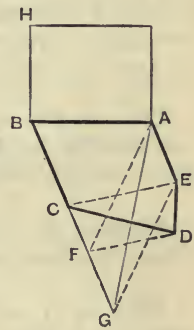
Construction 19.—'Draw a triangle, rectangle, parallelogram of given angle, or square equal in area to a polygon.'

If ABCDE is the polygon, produce a side BC, take an end diag. CE, draw DF parl. to CE;
∴ △ CEF = △ CED;
∴ quadl. ABFE = poln. ABCDE in area.

Thus, without altering the area, the number of sides is reduced by unity.

Similarly, take an end diag. AF of quadl. ABFE;
draw EG parl. to AF;
∴ △ ABG = quadl. ABFE = poln. ABCDE.

(ii.) On a side AB of tr. ABG, make AH a rect., or parm. of given angle, eql. to △ ABG;
∴ ▱ AH = poln. ABCDE.



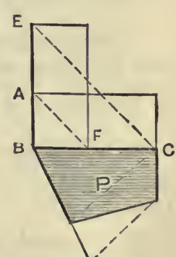
Similarly for a polygon of any number of sides. A square can be made eql. to rect. AH—i.e. to given polygon.

Construction 20.—'On a given side construct a rectangle, or parallelogram or triangle of given angle, equal in area to a given polygon.' *

On a side BC of poln. P construct rect., or parm. or tr. of given ang., AC, eql. to P.

Make BE the given side along BA, draw AF parl. to EC, draw parm. EF;
∴ ▱ EF = ▱ AC = P. (Th. 52.)

Or if △ ABC = P, △ EBF = ABC = P.



* This reduces to Constrs. 19 and 12.

In order to solve a problem, read the statement carefully, making a rough drawing of each part of the figure as soon as you understand the statement sufficiently. Compare different parts of the figure to find equal sides, angles, triangles, similar triangles, &c., and note the results of the comparison.

If this is not sufficient, draw auxiliary lines, circles, perpendiculars, bisectors of angles or sides, parallels, to see if any fresh relations can thus be discovered amongst the parts of the original figure; and search your text-book for theorems or constructions which may throw light on the problem.

In construction problems, start with the finished construction, and proceed as above, until you discover the key.

Do not make a triangle isosceles or equilateral, or a parallelogram rectangular or equal-sided, unless it is so given in the statement.

Appended are a few examples:

Theorem 55.—'The joins of mid points of sides of a triangle are parallel to the sides, and divide the whole triangle into four congruent triangles.' (Propl. division.)

Theorem 56.—'The sum of parallel sides of a trapezium is double the line bisecting the other sides.'

Theorem 57.—'The medians of a triangle are concurrent, and trisect each other.' (Simr. triangles.)

They meet in the **centroid** of the triangle.

Theorem 58.—'The bisectors of angles of a triangle are concurrent.'

They meet in the **incentre** of the triangle. (Th. 30.)

'The bisectors of exterior angles of a triangle are concurrent, two and two, with one of the bisectors of an interior angle.'

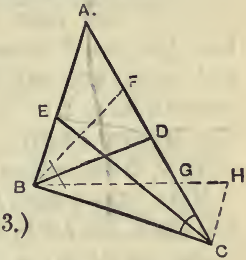
They meet in the three **ecentres** of the triangle.

Theorem 59.—'The three perpendiculars of a triangle are concurrent.' (Ang. in semicircle and in same arc, or simr. triangles.)

They meet in the **orthocentre** of the triangle.

The joins of their feet form the **pedal** triangle.

Theorem 60.—'If one angle B of a triangle ABC is greater than another C, the bisector of B is less than that of C.'



If BD, CE are the bisectors,

make ang. $ABF = CBH = \frac{C}{2}$, $BCH = B$.

Show triangle $CBH \equiv BCE$.

Compare BG and CE.

Show triangle $ABF \parallel\!\mid ACE$.

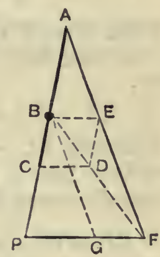
Compare BF and CE.

$BD <$ the greater of BG and BF. (Ex. XX. 23.)

Compare BD and CE.

Construction 21.—'If A, B, C are three points in order in a straight line, find a point P in it such that PB is a mean proportional between PA, PC.'

If P is the point, and PF, PG on another line $= PB, PC$, then $GF = CB$, and $\frac{PG}{PF} = \frac{PC}{PB} = \frac{PB}{PA}$;



$\therefore GB \parallel AF$.

Also, if $BE \parallel CD \parallel PF$,

$$\frac{CD}{PF} = \frac{CB}{PB} = \frac{GF}{PF};$$

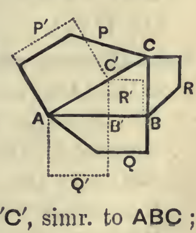
$\therefore CD = GF = BE$, and CDEB is a rhombus.

Hence the construction.

Make a rhombus BCDE, join BD, AE to F, draw FP parl. to BE or CD.

(Show that $PF = PB$, $GF = CB$.)

Pythagoras' Theorem Generalised.—'The sum $Q + R$ of the areas of two similar figures on the right-angled sides of a right triangle is equal to that of the similar figure P on the third side.'



If P, Q, R are first similarly situated, and then reduced to equal squares P′, Q′, R′ by Constr. 19, corresp. points and sides being used throughout the construction; then

side of P′ : side of Q′ : side of R′

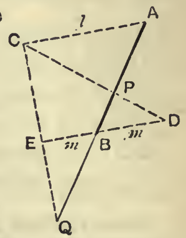
= side of P : corresp. side of Q : side of R;

$\therefore$ the sides of sqq. P′, Q′, R′ form a rt. tr. A′B′C′, simr. to ABC;

$\therefore P + Q = P' + Q' = R' = R$.

Construction 22.—'Divide a given line AB internally and externally in a given ratio μ.'

(i.) If the ratio is given as that of two lines l, m, and P is the internal point, and parls. AC, BD, BE eql. to l, m, m are drawn from A, B;
then tr. APC ||| BPD;
∴ C, P, D are collinear. Hence the constr.



Make AC $= l$, BD || AC and $= m$,
join CD to cut AB in P;
∴ AP : PB $= l : m = \mu$.

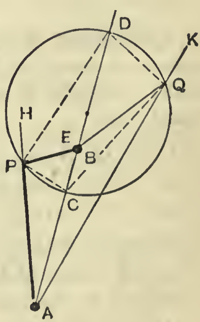
Similarly, make BE || AC and $= m$,
join CE to cut AB in Q;
∴ AQ : BQ $= l : m = \mu$.

(ii.) If the ratio is given numerically as μ, make BD, BE a unit and AC μ units, and proceed as before.

Note. Since P must be on CD, which cuts AB in one only point, it is clear that there is one only internal, and similarly one only external point.

Theorem 61.—'The locus of a point P whose distances from two fixed points A, B have a given ratio PA : PB $= \mu$ is the circle whose diameter divides AB in this ratio.'

If CD is the diamr. of this circle,
and CA : BC $= \mu =$ DA : DB;
then (i.) if P is on the locus,
PA : PB = CA : BC = DA : DB;
∴ PC, PD are bisectors of angs. APB, BPH;
∴ CPD is a rt. ang., and P is on the circle.



(ii.) If Q is a point on the circle,
and CQ bisects ang. AQE,
then DQ ⊥ CQ, and bisects ang. EQK;
∴ DA : DE = QA : QE = CA : EC;
∴ Alternando, DA : CA = DE : EC;
i.e. DE : EC = DA : CA = DB : BC;
∴ E coincides with B (see note, Constr. 22);
∴ QA : QB = CA : BC, and Q is on the locus.

Construction problems reduce generally to finding certain points by means of the loci on which they lie. The following summary will be useful.

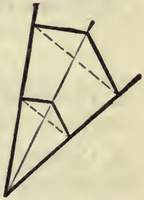
A. Locus a straight line.

(i.) Point whose join to a fixed point has fixed direction.

(ii.) Point equidistant from two fixed points.

Locus of vertex of isosceles triangle, given base; of centre of circle through two points; right bisector.

(iii.) Point equidistant from two lines; or whose distances in fixed directions from two lines have a given ratio.



Bisector of angle. Locus of vertex of triangle of given form (simr. to given triangle), its base resting on two given lines and parl. to fixed direction.

This is the general case of similarly situated figures or multiplication.

(iv.) Point dividing in given ratio any intercept of two parls., or any intercept of fixed direction between two lines.

(v.) Point at given distance from a line, or whose distance in fixed direction from a line is constant.

Locus of vertex of triangle or parm., given alt. and base line, or area, base, and base line.

This is the general case of translation.

(vi.) Point the sum of whose distances from two lines is constant. (Locus is sides of a rect., parl. to bisectors of angles of lines.)

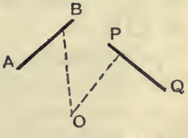
Locus of vertex of quadl., given sum of distances from two sides of given angle.

B. Locus a circle.

(i.) Point at given distance from fixed point.

Locus of any point of a figure turning about fixed point. Simple rotation.

Theorem 62.—'Any straight line AB can be rotated to coincide with an equal line PQ; and any polygon AB... to coincide with a congruent polygon PQ... of the same aspect.'



Rt. bisectors of AP, BQ determine the centre of rotation O.

This is the general case of congruence.

Note. A polygon of opp. aspect may be first reversed.

B. Locus a circle.

(ii.) Point whose joins to two fixed points form a given angle.

Locus of vertex of triangle, given base and opp. angle, of rt. triangle of given hypotenuse. (Constr. 9, 10 (viii.), Ch. II., or A, i., ii.)

(iii.) Mid point of chord of circle of given length, or through fixed point.

(iv.) Point whose distances from two fixed points have given ratio. (Th. 61.)

A, ii. is a limiting case of this.

EXAMPLES—XXII.

1. Construct a point P in a line PQ, whose join to a point A makes a given angle with PQ.

2. Construct an isosceles triangle, or circumcircle of triangle, given base and opp. ang. (A, i., ii., or B, ii.)

3. Inscribe in a triangle an equilateral triangle, also a triangle similar to a given triangle, given one side parl. to fixed direction. (A, iii.)

4. Construct a triangle, given base, area, median from one end of base. (A, iv., v., B, i.)

5. Find locus of diagonal point of a trapezium, one parallel side fixed, the other and its alt. given in magnitude. (A, iv.)

6. Construct a triangle, given one angle in position, and sum of distances from sides of foot of perpendicular from vertex of angle on opposite side. (A, vi., B, ii.)

7. Construct quadl. ABCD, given angle C in position, triangle ABD in form, direction of BD, sum of distances of A from sides of ang. C. (A, iii., vi.)

8. Construct a triangle, given base, opp. ang., and alt. (A, v., B, ii.); given base, opp. ang., and area (A, v., B, ii.); given base, alt., ratio of sides (A, v., B, iv.); given base, opp. ang., ratio of sides (B, ii., B, iv.).

9. Inscribe a triangle in a circle, given one side in length, opp. vertex, and distance of another side from centre (B, i., iii.); given one side in length, a point on another side, and distance of this side from centre (B, i., iii.).

10. Inscribe a triangle in a circle, given one side fixed in position and the ratio of the other sides. (B, iv.)

Suggest a simpler alternative construction.

11. If one end of a straight line of given length and direction moves in any path, the other end describes a congruent path. (A, v.)

EXAMPLES—XXIII.

THEOREMS.

1. A diagonal of a quadrilateral bisects the angle between two equal sides; show that the other sides are equal.

2. Equilateral triangles ABD, BCE are formed externally on two sides of a triangle ABC; show that DC = AE.

3. In a regular hexagon the triangle formed by joining three alternate vertices is equilateral.

4. In a regular pentagon ABCDE, show that the triangle ACD has the angles C, D each double of A. (Calc. in degrees ang. of pentagon and hence of triangles ABC, ACD.)

5. If the bisector of angle A of a triangle coincides with the median, the triangle is isosceles.

6. Every rhombus is a parallelogram.

7. A quadrilateral is a parallelogram when (i.) its diagonals bisect each other, (ii.) opposite angles are equal, (iii.) opposite sides are equal.

8. A parallelogram whose diagonals bisect its angles is a rhombus. Is this true of any quadrilateral?

9. How can you distinguish parallelogram, rhombus, rectangle, square by their diagonals?

10. The total height of a staircase is the sum of heights of the separate steps.

11. If AB, CD are equal chords of a circle (the order on the circle being ABDC), the triangles ABD, CDB are congruent. (Use symmetry.)

12. Squares ABDE, ACFG are described on two sides of a triangle; show that the median AX of the triangle is perpendicular to and the half of EG.

13. Show also in Ex. 12 that BG is perpendicular and equal to CE.

14. If the non-parallel sides of a trapezium are equal, two opposite angles are supplementary.

15. Medians AX, BY of a triangle are produced to double their lengths at F, G; show that FG passes through C.

16. One straight line AB from a point A to a circle, centre O, is greater than another AC, if the angle AOB at the centre is greater than AOC.

17. If two sides of a parallelogram are given in magnitude, their diagonal increases as their angle diminishes. What are its greatest and least values?

18. If a straight line bounded by two parallels is bisected at a point O, any other straight line through O bounded by them is bisected at O.

19. The joins of mid points of sides of a quadrilateral form a parallelogram.

20. Parallels to the equal sides of an isosceles triangle are drawn from a point in the third side; show that the perimeter of the parallelogram thus formed is the sum of the equal sides.

21. M is the mid point of a line AB. The sum of perpendiculars from A and B to a line cutting AB produced is twice the perpendicular from M.

22. The diagonals of a trapezium divide each other proportionally.

23. A parallel to the parallel sides of a trapezium divides the other sides proportionally.

24. If two diagonals of a quadrilateral divide each other proportionally, the figure is a trapezium.

25. AD is the bisector of angle A of a triangle ABC, M the mid point of BC, and BE, CF perpendiculars to AD. Show that ME=MF.

26. The lines BE, DF drawn to the mid points E, F of the sides AD, BC of a parallelogram, trisect the diagonal AC.

27. If DE parallel to BC in triangle ABC meets AB, AC in D, E, then BE and CD divide each other proportionally.

28. In Ex. 27 any line through A divides DE, BC proportionally.

29. If parallels APB, CQD are divided proportionally at P, Q, the joins AC, PQ, BD are concurrent.

30. The non-parallel sides of a trapezium and the line through the mid points of the parallel sides are concurrent.

31. The projections of two parallel lines on a given line are proportional to the lines.

32. If two triangles have their angles respectively equal, the sides of one are either all greater than, or all less than, or all equal to the corresponding sides of the other.

33. The circumradii of similar triangles are proportional to the sides.

34. If three straight lines OAB, OCD, OEF are similarly divided, the triangles ACE, BDF are similar.

35. If two similar triangles are placed with corresponding sides parallel, the joins of corresponding points are concurrent. Is this true of similar polygons?

36. If ABCD, EFGH are similarly divided lines, the triangles of their parts are similar.

37. A parallelogram having an angle A common with a parallelogram ABCD, and the opposite vertex E on the diagonal AC, is similar to it.

38. If two corresponding sides of two similar parallelograms having a common angle are in a straight line, their diagonals through this angle coincide.

39. If the pairs of opposite sides of a parallelogram cut proportionally the parallels to the other sides through a point P, then P is on a diagonal.

40. If P, Q divide the sides AD, BC of a parallelogram so that PA : PD = QC : QB, then PQ bisects the diagonals. (Use alternando.)

41. If DE parallel to BC meets AB, AC in D, E, and if BE, CD meet in O, and OF parallel to BC meets AB in F; then ADFB is divided harmonically.

42. If P is any point in a diameter AB of a circle, PC any line to the circle, Q a point on AB produced such that ang. ACQ = ACP, show that QAPB is divided harmonically.

43. The lines joining the ends of parallel diameters of two circles divide the join of centres harmonically, in the ratio of the radii.

44. If M is the mid point of a line AB, divided harmonically at P, Q, show that $MP . MQ = MB^2$. (Treat as algebraic products.)

45. If in a right triangle ABC, B the right angle, AB is half BC, then AC – AB is the mean part of BC. (Constr. 15, Note i.)

46. If in Th. 44 CD is produced to F so that CF = CA, show that AF = FB = BC, and BF ∥ AC.

47. The sum of perpendiculars to the sides of an equilateral triangle from a point inside it is equal to an altitude.

48. In a square ABCD, Q is the mid point of CD, and AP meeting CD in P is equal to CP + CB. Show that the angle BAP is double of QAD.

49. A triangle is isosceles when (i.) two altitudes are equal, (ii.) two medians are equal, (iii.) two bisectors of angle are equal.

50. One side of a triangle is greater than another when (i.) its altitude is less, (ii.) its median is less, (iii.) its bisector of angle is less than that of the other.

51. The four triangles formed by the half diagonals of a parallelogram with the sides are equal in area.

52. Any straight line through the diagonal point of a parallelogram bisects its area.

53. If triangles ABC, A′B′C′ have $a = a'$, $b = b'$, C = supplement of C′, their areas are equal.

54. If straight lines AB, CD intersecting at E make the triangles AEC, BED equal in area, then BC ∥ AD.

55. If G is the centroid of the triangle ABC, the areas GAB, GBC, GCA are equal.

56. If ABC, DBC are triangles equal in area on opposite sides of BC, then AD is bisected by BC.

57. If G is a point in a triangle such that areas GAB, GBC, GCA are equal, G is the centroid.

58. If the diagonal AC of a quadrilateral ABCD is bisected by BD, the triangles ABD, CBD are equal in area.

59. A straight line through a vertex D of a parallelogram ABCD cuts BC, AB in F, G; show that ABF, CFG are equal in area.

60. If AD is the perpendicular on BC in triangle ABC, H the orthocentre, show that rect. DA . DH = rect. DB . DC.

61. In a trapezium the rectangle of two of the parts into which the diagonals divide each other is equal to that of the other two.

62. If AD, BE are perpendiculars of a triangle ABC, show that rect. CD . CB = rect. CE . CA. If X, Y are mid points of BC, CA, show also that CX . CD = CY . CE.

63. The rectangle contained by one of the equal sides of an isosceles triangle, and the projection on it of the third side, is half the square of this side.

64. If BD, CE are drawn from the ends of the hypotenuse of a right triangle to the sides, $BD^2 + CE^2 = BC^2 + DE^2$.

65. If a point P is joined to the points of a rectangle ABCD, then $PA^2 + PC^2 = PB^2 + PD^2$.

66. If BC is the hypotenuse of a right triangle, BC^2 is four-fifths of the sum of squares of medians from B, C.

67. A square has a greater area than a rectangle of the same perimeter. (Constr. square equal to rectangle.)

68. A square has a less perimeter than a rectangle of the same area.

69. Of all parallelograms of given area, the square has the least perimeter.

70. Of all parallelograms of given perimeter, the square has the greatest area.

71. Can you prove Exx. 69 and 70 if quadrilateral is written for parallelogram?

72. A square ABCD and rectangle AEFG have a common angle A, and the point E is on AB. If their areas are equal, show that ED $\parallel$ BG.

73. Conversely, if in Ex. 72 ED $\parallel$ BG, show that the areas are equal.

74. Assuming that angles in the same arc of a circle are equal, show that if AB, CD are two chords of a circle meeting in P, the rect. PA . PB = rect. PC . PD.

75. Using Ex. 74, show that if a chord CD of a circle is bisected at M by a diameter AB, then rect. AM . MB = MC^2.

76. Show also in Ex. 75 that AC^2 = AM . AB.

77. If P is any point on a chord of a circle, centre O, bisected at M, show that $PO^2 = PM^2 + MO^2$.

78. Show that the sum of squares on the right-angled sides of a right triangle is four times the square on the median of the hypotenuse.

EXAMPLES—XXIV.

CONSTRUCTIONS.

1. On two sides of an angle A find points B, C such that BC is equal in length to one line D, and parallel to another E.

2. Find a point P in the side BC of a triangle such that if PQ, PR are perpendicular to AB, AC, then AQ = AR.

3. From a given point P draw lengths PQ, PR to two parallel straight lines so that PQ and PR are equal and perpendicular to each other.

4. Construct a triangle, given in position the mid points of sides.

5. Construct a triangle, given in position the feet of perpendiculars.

6. Measure the angle of elevation of the sun when a post 12 ft. high throws a shadow of 24 ft.

7. Draw an elevation of a staircase, height of each step 9″, depth 12″, to show 10 steps, half an inch to the foot.

8. A tower stands on the edge of a cliff 100 ft. high. From a point below the cliff the elevation of the base of the tower is 45°, and of the top 60°. Find by diagram the height of the tower.

9. Two posts A, B on one bank of a river are 100 ft. apart. Two others C, D on the same side, each 200 ft. from the bank, are 500 ft. apart. The joins AC, BD just cover a post P on the opposite bank. Find the width of the river.

10. Inscribe a rhombus in a parallelogram, having one vertex at a fixed point on one side of the parallelogram.

11. Construct a quadrilateral ABCD, diagonal BD = 2·3 cm., AB = AD = 2·7 cm., CB = CD = 3·2 cm. Draw the right bisector of BD.

12. Draw a parallelogram, sides $1\frac{3}{16}''$, $2\frac{3}{8}''$, angle 60°. Is the diagonal through this angle greater or less than that of the rectangle of the same sides? Why? Measure the two.

13. Given a finite straight line AB, construct a continuation of it CD, separated from AB by an obstacle.

14. Construct a fourth proportional to 1″, 2·07″, 2·44″, and measure its length.

15. Show that the result in Ex. 14 represents the product of 2·07 and 2·44. Hence deduce a method of representing a product geometrically.

16. Two finite straight lines ABCD and PQ, not parallel, are given in position. Divide PQ similarly to ABCD.

17. Draw a line between two sides of an angle to pass through a fixed point and to be divided in a given ratio at that point.

18. Take any point P inside an equilateral triangle, 3 cm. side. Draw a straight line terminated by the sides and divided at P in the ratio 3 : 7.

19. Construct the locus of a point P whose distances from two given lines OA, OB at an angle of 63° are as 4 : 5.

20. Draw a parallelogram, sides 2·8 cm., 1·9 cm., angle 54°. Construct a similar figure, with sides greater in the ratio 12 : 7.

21. Construct a regular pentagon (use protractor) in a circle of 2 cm. radius. Construct a similar figure whose diagonals are half as long again as those of the first.

22. Construct the locus of a point dividing a line between two parallels $1\frac{1}{2}''$ apart in the ratio 3 : 5.

23. AB, CD are straight lines which meet if produced outside the paper. Through a given point P draw a straight line to their point of intersection. (Use parallel lines similarly divided, or similarly situated triangles.)

24. Draw also the bisector of angle of two such lines as AB, CD in Ex. 23.

25. Construct a triangle the sum of whose sides is 3″, and two of whose angles are 52°, 48°.

26. Draw a square whose side is the mean proportional of 1·8 cm. and 3·2 cm. How does it compare in area with the rectangle of these sides?

27. Give a geometrical construction to find $\sqrt{7}$ to two decimal places. Is your value correct? (Use $\frac{1}{2}$-inch diagonal scale.)

28. Take $AB=\frac{3}{4}''$, $BC=\frac{1}{2}''$ in a straight line, and find P in BC produced so that PB is mean proportional of PC, PA. Measure PC. (Constr. 21.)

29. Take $AB=\frac{3}{4}''$, $BC=\frac{1}{2}''$, as in Ex. 28, and construct AD the mean proportional of AB, AC. Compare DC with PC in Ex. 28.

30. Divide a straight line 2·6″ long in the ratio 9 : 4, and find the mean proportional of the parts.

31. Find points D, E in the sides BC, CA of a triangle so that DE parallel to AB is the mean proportional of BD, DC.

32. Draw a straight line ABC, AB = 2·7 cm., BC = 1·8 cm.; find a point D so that ABCD is divided harmonically.

33. Draw an isosceles triangle, $a=b=2{\cdot}9$ cm., C = 30°. Draw CD the bisector of angle C. Can you find a point E so that ADBE is divided harmonically? Why? How do you interpret the result?

34. If AP, AQ, AR are drawn from A to the opposite side BC of a parallelogram, bisecting BC, BP, BQ respectively and meeting the diagonal BD in H, K, L; find the ratio BL : BD.

35. Divide AB, = 1·4″, harmonically with internal parts in the ratio 3 : 4.

36. Construct and calculate numerically the mean part of a line 1″ long.

37. Calculate from Construction 15 the ratio of a line to its mean part (i.) as a surd, (ii.) as a decimal.

38. If in a right triangle ABC, B the right angle, AB=2BC, and BD is perpendicular to AC, find the ratio AD : DC.

39. Construct an isosceles triangle ABC such that the side BC, 1″ long, is the mean part of the equal sides AB, AC. Bisect angles B, C by BD, CE each equal to AB. Prove that AEBCD is a regular pentagon.

40. Find the locus of a point P dividing in a given ratio, say 5 : 3, the join of a fixed point V to a point Q moving on a given circle.

41. Construct the locus of P when PA : PB = 3 : 8, AB = 3·3 cm., A and B being fixed points.

42. Construct the locus of the point A of a triangle when a=2·3 cm., $b : c$=3 : 5.

43. Construct a triangle, a=2·5 cm., $b : c$=3 : 2, C=30°. Can it be done if C=80°?

44. Construct by ruler and compass only an angle of 27°. (Constr. 16. Bisect ang. A twice.) Test by protractor.

45. Draw a triangle, a=1·12″, b=1·28″, c=2·08″. Inscribe in it an isosceles right triangle with the hypotenuse parallel to the bisector of angle A.

46. Calculate the areas: (i.) triangle, base 2·7 cm., alt. 3·2 cm.; (ii.) trapezium, parl. sides 1·9 cm., 2·9 cm., alt. 1·3 cm.; (iii.) right triangle, perp. sides 12·7 cm. and 33·8 cm.

47. Calculate the sides of the equivalent squares in Ex. 46.

48. Draw a regular hexagon on a side of 2·5 cm., and draw a triangle of equal area. What is the area?

49. Construct a rhombus equal in area to a given parallelogram.

50. Construct a square equal to the (i.) sum, (ii.) difference of squares on sides of 2·5 cm. and 1·8 cm.

51. A pentagon ABCDE represents an estate ($\frac{1}{8}$″ to the mile). AC = $1\frac{7}{8}$″; altitudes of B, D, E from AC, 1″, $1\frac{3}{8}$″, $1\frac{1}{4}$″; projections of AB, AE, AD on AC, $\frac{1}{2}$″, $\frac{3}{4}$″, $1\frac{3}{8}$″. Draw a plan and calculate the area. (This is the ordinary surveyor's method of measuring the area of a field.)

52. ABCD (4″ to the mile) represents a field; B, D are right angles; AC=3·2″; altitudes of B, D from AC, 1·3″, 1·1″. Draw a plan and calculate the area.

53. Calculate the areas: (i.) equil. triangle, side 3·2 cm.; (ii.) parm., base 2·2 cm., alt. 3·8 cm.; (iii.) isosceles rt. triangle, hyp. 1·41″.

54. Construct a quadl., AC=2·7 cm., AB=BC=2·3 cm., CD=DA =2·9 cm. Describe an equal isosceles right triangle.

55. On a scale of $\frac{1}{100}$″ to the yard represent a square field of 10 acres.

56. Make a square equal to a rectangle of sides 1·8″, 5″.

57. Divide a straight line of 2″ into two parts whose rectangle is equal to the square on $\frac{3}{4}''$ side.

58. Given one side 2·74″ of a rectangle equal to a square on 1·54″ side, construct the other side.

59. Construct the side of a square of 3·76 sq. in. Measure.

60. Construct an equilateral triangle of 3·7 cm. side, and construct a square of the same area.

61. Construct a mean base triangle (Constr. 16), base $1\frac{1}{2}''$, and make a square equal to it.

62. Inscribe a given square in a given square. (Constr. 10, vii.)

63. Calculate the diagonal of a square of 5·96 sq. in. area.

64. Construct a triangle, $a=1·32''$, $b=2·75''$, $c=1·96''$. On the other side of a construct a triangle of equal area, having one of its sides 2″.

65. Construct a rectangle, diagonal 2″, equal to the triangle of Ex. 64.

66. Draw a rectangle, sides 3·6 cm., 4·8 cm., and construct an equal rectangle, one side 3·2 cm.

67. Given a line AB, 1″, and a point P in BA produced, $\frac{1}{2}''$ from A; find points D, D′ such that the square on PD or PD′ is equal to the rectangle PA . PB.

68. Given a line AB, 4 cm., find the points C, D dividing AB internally and externally in the ratio 5 : 3.

Find also the point P such that PC is the mean proportional of PA, PB.

69. Set off a line AB, 1″. Make AP perpendicular and equal to AB. Along AB from A set off in succession lengths AC equal to PB, AD to PC, AE to PD, and so on.

Show that AB, AC,... represent the complete series of quadratic surds of whole numbers $\sqrt{1}$, $\sqrt{2}$, $\sqrt{3}$...

Which of this series of surds would be the simplest to pick out as a check on the accuracy of your work?

70. Take a line AB, 1″. Make BC, perpendicular to AB, 1″; CD, perpendicular to AC, 1″; DE, perpendicular to AD, 1″, and so on.

Show that the series of lines AB, AC, AD, AE represents the same series of surds as those of Ex. 69, and that the two constructions are really the same.

71. Construct an isosceles right triangle, side 1″. On the hypotenuse construct a second, and on its hypotenuse a third, and so on, until a hypotenuse is constructed in line with the original base produced. Measure this.

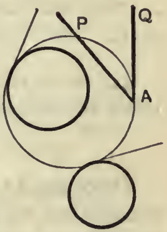
72. On a base of 1 cm. construct a mean base triangle (Constr. 16). On its side construct another, and so on until the side of the fifth triangle is obtained along the original base. Measure this side.

PART II.

CHAPTER IV.

THE CIRCLE—CHORD, TANGENT, ANGLE, AND RECTANGLE PROPERTIES—EXPERIMENTAL SOLID GEOMETRY.

Definition 28.—A **secant** or **transversal** of a circle through any point in its plane is a straight line through the point cutting the circle.



The **tangent** of a circle at a point A on it is the secant or transversal AP through A whose second point P on the circle coincides with A.

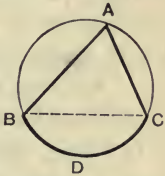
(Turn a straight-edge about a pin fixed at A, until P is as near A as possible.)

In Th. 64 it is shown that there is one only tangent at each point of the circle.

Note. The *length* QA from a point Q on a tangent to the point of contact A is often called the tangent from Q to the circle.

Definition 29.—Two circles **touch** which have a common tangent at a point where they meet.

Definition 30.—An **angle in a circle** is an angle whose point is on the circle, and whose sides cut the circle.



It *stands on* the arc cut off the circle by its sides; and it *is in* the arc or segment containing the point of the angle and terminated by the sides.

The angle A *stands on* arc BDC, and *is in* arc or segment BAC.

Definition 31.—A **segment** of a circle is a part of a circle bounded by a chord and one of its arcs.

The two arcs or segments into which a chord divides a circle are **opposite** or **alternate.**

Definition 32.—A **polygon** whose vertices are on a circle is **inscribed** in the circle; and the **circle** is then **circumscribed** to the polygon.

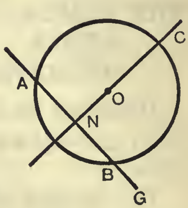
Definition 33.—A **polygon** whose sides touch a circle is **circumscribed** to the circle; and the **circle** is then **inscribed** in the polygon.

Theorem 63.—'The centre of a circle lies in any right bisector of a chord; and a chord meets the circle in two only points.'

If NC is the rt. bisector of chd. AB of a circle, centre O;

(i.) O is equidistant from AB;

∴ O lies in rt. bisector NC.



(ii.) If G is any third point in AB, the rt. bisector of AG || NO, and therefore does not traverse O;

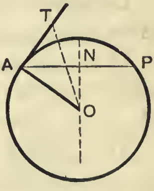
∴ G is not on the circle, centre O;

∴ chd. AB meets the circle in two only points A, B.

Ex. Show that two chords of a circle, not both diameters, cannot bisect each other.

Theorem 64.—'The tangent at any point on a circle is perpendicular to the radius of the point, and meets the circle at its point of contact only.'

If A is a point on a circle, centre O, P any other point on the circle; draw ON, rt. bisector of AP.



Then (i.) if P moves along circle to coincidence with A, the rt. bisector ON coincides with OA, and sect. AP coincides with tangt. AT;

∴ tangt. AT $\perp$ rad. OA.

(ii.) Hyp. OT > OA in rt. triangle OAT;

∴ T is outside the circle;

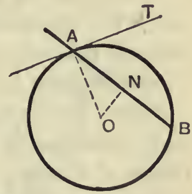
∴ A is the only point of the tangent on the circle.

Note. The tangent is entirely outside the circle except at A; hence no tangent can be drawn from a point inside a circle.

Ex. 1. Show that the circle on OA as diameter touches AT and also the first circle.

Ex. 2. If TO meets the circle on OA as diameter in M, show that TA is the mean proportional of TN, TO.

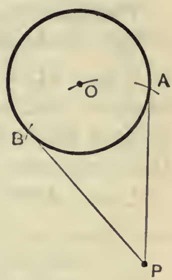
Theorem 65.—'Every straight line, except a tangent, in the plane of a circle and meeting it, cuts the circle twice.'



If a str. line AN meets a circle, centre O, at A;
make ON perp. to AN, and NB eql. to NA;
∴ NO is rt. bisector of AB;
∴ OB = OA, and B is on the circle.
But ON < OA in rt. triangle OAN;
∴ N is inside the circle, and AB passes from inside to outside in crossing the tangents at A, B.

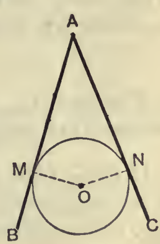
Cor.—'A chord of a circle lies entirely inside the circle.'

Theorem 66.—'Two only tangents can be drawn to a circle from an outside point in its plane; these are equal in length, and symmetrical about the join of the point to the centre.'



If P is a point outside a circle, centre O, and PA is tangt. at A;
then PAO is a rt. ang.
∴ A is on circle, diam. OP, which cuts the first circle in two only points A, B;
∴ there are two only tangts. PA, PB;
and the whole figure is symmetrical about PO;
∴ the tangts. PA, PB are equal,
and symmetrical about PO.

Theorem 67.—'The locus of the centre of a circle touching two straight lines is the bisectors of their angles.'

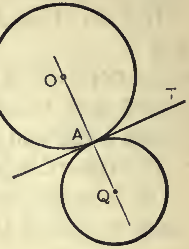


If O is the centre of a circle touching AB, AC at M, N,
perp. ON = OM;
∴ O is on a bisector of BAC. (Th. 30.)
And any point on the bisectors is equidistant from the lines, and may be a centre.

Ex. Show that A is on the bisector of ang. NOM.

Ex. Construct a centre of a circle to touch three straight lines.

Theorem 68.—'Two circles which touch meet on their line of centres, and meet in one only point.'



If two circles, centres O, Q, touch at A, and AT is the common tangt., OA and QA each $\perp$ AT;

$\therefore$ OA, AQ are in a straight line.

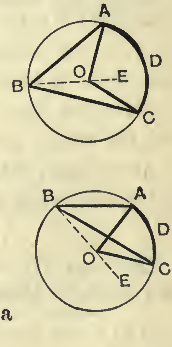
Also, the common chd. from A $\perp$ OQ, and coincides with AT, which meets the circles at A only;

$\therefore$ the two circles meet at A only.

Note. The join of centres of two circles which touch is the sum or difference of their radii.

Ex. Show that two circles do not meet at all if the join of their centres is greater than the sum or less than the difference of the radii.

Theorem 69.—'An angle in a circle is half the angle at its centre on the same arc.'



If ang. AOC at centre O of a circle is on same arc ADC as ang. ABC in circle, produce BO to E;

then ang. OBA = OAB, $\because$ OA = OB;

$\therefore$ ang. AOE = suppt. of AOB

= OBA + OAB, in tr. AOB,

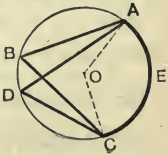
= 2OBA.

Similarly, ang. COE = 2OBC;

$\therefore$ sum or diffce. ang. AOC = 2ABC.

Ex. Show by this method that the angle in a semicircle is a right angle.

Theorem 70.—'Two angles in a circle in the same arc or segment are equal.'



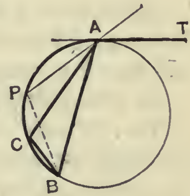
If ABC, ADC are angs. in same arc or segment of a circle, centre O,

ang. ABC = $\frac{1}{2}$AOC, on same arc AEC,

= ADC.

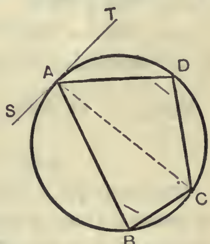
Theorem 71.—'The angle of a tangent to a circle and a chord through the point of contact is equal to the angle in the opposite arc, or alternate segment.' *

If TAB is an angle of a tangt. AT and chd. AB of a circle, and C, P are points in the opposite arc;
move P round the arc to A;
then as P moves to coince. with A,
PA " " tangt. AT,
ang. BPA " " ang. BAT.
But ang. P always = C, same arc;
∴ ang. TAB = ACB in opp. arc.



Theorem 72.—'Angles in opposite arcs of a circle are supplements; or, opposite angles of a cyclic quadrilateral are supplements.'

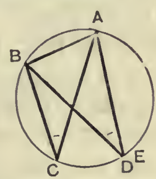
If B, D are angs. in opp. arcs ABC, ADC of a circle, draw tangt. SAT;
∴ ang. TAC = B in opp. arc;
similarly, ang. SAC = D "
∴ ang. B = TAC = suppt. of SAC
= suppt. of D.



Ex. Show this also by means of angles at centre.

Theorem 73.—'If equal angles C, D stand on the same line AB, on the same side, the points A, B, C, D are concyclic.'

Draw circ. ABC, cutting BD in E;
∴ ang. AEB = C, same arc, = ADB;
∴ AE ‖ AD and coincides with it;
∴ point D coincides with E;
i.e. D is on circle ACB.



Cor.—'The locus of the vertex of a given angle, whose sides traverse fixed points, is two equal arcs of circles.'

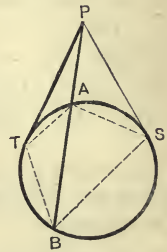
Theorem 74.—'If angles C, D standing on the same line AB, on opposite sides, are supplements, A, B, C, D are concyclic.'

Draw circle ACB cutting BD in E; prove as in Th. 73.

* Illustrate by angle of set-square at P, and pins at A, B.

Theorem 75.—'A tangent from a point to a circle is the mean proportional of the parts of any secant from the point; and a line from a point to a circle, mean proportional of the parts of a secant from the point, is a tangent.'

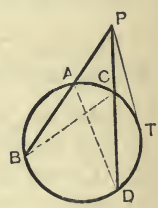
(i.) If PT is a tangt., PAB a secant from point P to a circle; then ang. PTA = PBT, in opp. arc,
and ang. P is common to trs. PTA, PBT;
∴ tr. PTA ||| PBT;
∴ PA : PT = PT : PB;
i.e. tangt. PT is mean propl. of PA, PB, parts of secant PAB.



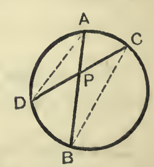
(ii.) If PS is mean propl. of PA, PB;
then PA : PS = PS : PB,
and ang. P is common to trs. PSA, PBS;
∴ tr. PSA ||| PBS;
∴ ang. PSA = PBS, in opp. arc;
∴ SP coincides with the tangt. at S.

Note. Sq. on PT = rect. PA . PB = sq. on PS.

Theorem 76.—'The rectangle of the two parts of any chord or secant from a point to a circle is constant, and equal to the square on the tangent from the point.'*



If PCD is a fixed secant or chd. from P to a circle, PAB any chd. or secant whatever, PT tangt. from P;*
then in trs. PBC, PDA,
ang. P is common, ang. B = D, same arc;
∴ tr. PBC ||| PDA;
∴ PB : PD = PC : PA;
∴ rect. PA . PB = PC . PD = const.
= PT^2, if PAB moves to coincidence with PT.



Ex. Ptolemy's Theorem.—The sum of rectangles of opposite sides of a cyclic quadrilateral is equal to the rectangle of diagonals.

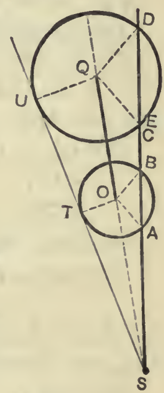
* When the point is outside.

Definition 34.—The **centres of similitude** of two circles are the two points dividing the join of centres in the ratio of the radii.

They are constructed (Constr. 22, Ch. III.) by joining ends of parallel diameters of the circles.

If a secant through a centre of similitude of two circles cuts them in points A, B and C, D, two nearest or two farthest points A, C or B, D are **corresponding points,** and one nearest and one farthest, A, D or B, C, **inverse points.** (See fig. below.)

Theorem 77.—'The radii from corresponding points of a secant through a centre of similitude of two circles are parallel.'



If SABCD is a secant from centre of simde. S of circles, centres O, Q;
make QE parl. to OA, to meet SD in E;
∴ QE : OA = QS : OS = QC : OA;
∴ QE = QC, and E is on circle CD;
i.e. E coincides with C;
∴ QC ∥ OA; similarly, QD ∥ OB.

Cor.—'The common tangents of two circles pass through the centres of similitude.'

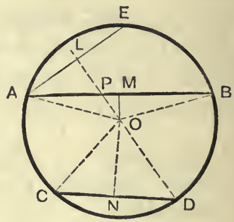
For if SAD turns about S until the points A, B coincide at T, the points C, D will also coincide at U.

Theorem 78.—'The rectangle of the parts of a secant of two circles from a centre of similitude to two inverse points is constant.'

In the above figure, if STU is a common tangent;
then SB : SD = SO : SQ = ST : SU;
∴ tr. SBT ||| SDU, and ang. SBT = SDU.
But ang. STA = SBT, opp. arc, ∵ ST is tangt.;
∴ tr. STA ||| SDU;
∴ SA : ST = SU : SD;
∴ rect. SA . SD = ST . SU = const.

Note. TB ∥ UD, and we can prove in like manner that any two corresponding chords of the circles are parallel. Thus the circles are similarly situated with respect to their centres of similitude.

Theorem 79.—'A greater chord of a circle is nearer the centre, and subtends a greater angle at the centre, than a less chord; and equal chords are equidistant from the centre and subtend equal angles at it.'



If AB, CD are chds. of a circle, centre O, and OM, ON bisect angs. AOB, COD, and therefore also are right bisectors of chds. AB, CD;
turn fig. OCND round in the plane into the position OELA.

Then if ang. AOB > COD, i.e. > AOE, the half AOM > AOL;
∴ the compt. OAM < compt. OAL;
∴ OL passes first inside, and then outside the triangle OAM, and cuts AM in a point P;
∴ OM < hyp. OP, in rt. tr. OPM,
< OL
< ON.

Similarly, AM > AP > AL;
∴ the double AB > AE
> CD;
i.e. (i.) if ang. AOB > COD,
chd. AB > chd. CD, and dist. OM < dist. ON.

Similarly (ii.), if ang. AOB < COD,
chd. AB < chd. CD, and dist. OM > dist. ON.

Also (iii.), if ang. AOB = COD = AOE,
E coincides with B, and L with M;
∴ chd. AB = AE = chd. CD, and dist. OM = OL = dist. ON.

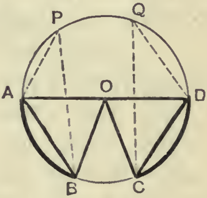
Cor.—'Equal chords from a point on a circle, on the same side of the diameter through the point, coincide.'

Ex. 1. Show by congruent right triangles that equal chords are equidistant from the centre; and conversely, that chords equidistant from the centre are equal.

Ex. 2. Find the greatest chord of a circle. What angle does it subtend at the centre?

Theorem 80.—'The arcs and chords in a circle of equal angles at centre or circumference are equal; the angles and chords of equal arcs are equal; the angles and arcs of equal chords are equal.'

If AOB, COD are angles at the centre O of a circle,
AB, CD their chds. and arcs;
turn sector COD about O in the plane so that OC coincides with OA,
and arc CD lies along AB.



Then (i.), if ang. AOB = ang. COD,
OD coincides with OB, and D with B;
∴ arc AB = arc CD, and chd. AB = chd. CD.
Also, if ang. APB = CQD,
the double AOB = COD, &c., as before.
(ii.) If arc AB = arc CD,
D coincides with B, and OD with OB;
∴ ang. AOB = ang. COD, and chd. AB = chd. CD,
and half ang. APB = half ang. CQD.
(iii.) If chd. AB = chd. CD,
ang. AOB = ang. COD. (Th. 79.)
∴ also half ang. APB = half ang. CQD, and arc AB = arc CD.

Cor. (i.).—'Two sectors AOB, COD in a circle are congruent which have equal angles, equal arcs, or equal chords.'

Cor. (ii.).—'If angles in equal circles are equal, their arcs, chords, and sectors are equal, and similarly for arcs, chords, or sectors.'

Ex. 1. Show by congruent triangles that equal chords of a circle subtend equal angles at the centre, and hence that they cut off equal arcs.

Ex. 2. Show that AD, BC are parallel. Also that AB, DC intersect on the line bisecting AD, BC.

Ex. 3. If AB and DC meet in E, show that EB = EC.

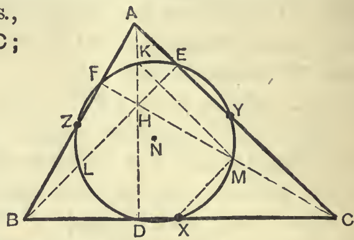
Ex. 4.* The bisector of an angle in a circle bisects the opposite arc.

Ex. 5. The bisectors of angles in a given arc of a circle are concurrent.

* Very important.

Theorem 81.—'The circle through the mid points of sides of a triangle passes also through the feet of perpendiculars and bisects the joins of the orthocentre to the vertices.' (N-circle, or nine-point circle.)

If X, Y, Z are mid points of sides of a triangle ABC, H the orthocentre, D, E, F feet of perps., K, L, M mid points of HA, HB, HC;



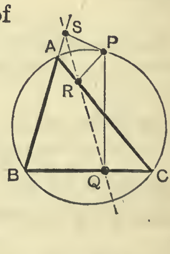
then XM bisects BC, HC, and || BE;
similarly, MK || AC;
∴ ang. XMK = BEA = a rt. ang.;
∴ the circle on diam. XK passes through M, and simly. through L, and it also passes through D;
∴ the circle through K, L, M passes through X and D, and similarly through Y and E, and Z and F;
i.e. the circle through X, Y, Z passes through D, E, F and K, L, M.

Note. The angle XYD in the arc on XD = B − C. (B > C.)
For ext. ang. CXY = int. opp. angs. XYD + YDC, in tr. XYD;
i.e. ang. B = XYD + C, since medn. YD = YC, in rt. tr. ADC;
∴ XYD = B − C.

Ex. Find in the same way the values of angles YZE, ZXF.

Theorem 82.—'The feet of perpendiculars to the sides of a triangle from a point on its circumcircle are collinear,' (Simson line.)

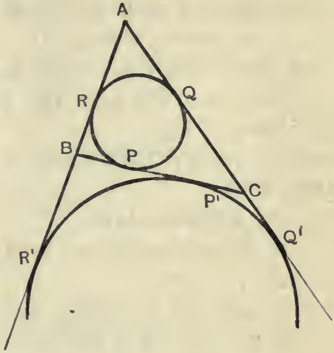
If PQ, PR, PS are perps. to the sides of triangle ABC from P on circumcircle;
PSAR and PRQC are cyclic quadls.;



∴ ang. PRS = PAS, in same arc,
= suppt. of PAB
= PCB (circumcircle)
= suppt. of PRQ, opp. arc;
∴ SR, RQ are in a straight line.

Ex. If from a point P on the circumcircle of a triangle, lines PQ, PR, PS to the sides make equal angles with these in order, then Q, R, S are collinear.

Theorem 83.—'The tangents from the vertices A, B, C of a triangle to the incircle and the ecircle of angle A are $s-a$,* $s-b$, $s-c$, and s, $s-c$, $s-b$, respectively.



If P, Q, R and P′, Q′, R′ are points of contact of the sides, then

(i.) the six tangts. of incle. $=$ sum of sides $= 2s$;

$\therefore$ AR + BP + CP = AQ + BR + CQ $= s$;

and BP + CP $= a$;

$\therefore$ AR $= s - a =$ (similarly) AQ.

Similarly, BP = BR $= s - b$,

CQ = CP $= s - c$.

(ii.) AR′ = AB + BR′ = AB + BP′,

AQ′ = AC + CQ′ = AC + CP′;

$\therefore$ AR′ + AQ′ = AB + AC + BC $= 2s$;

$\therefore$ AR′ = AQ′ $= s$,

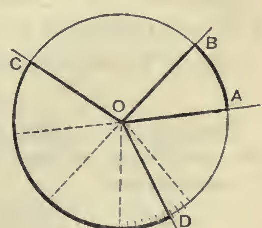
and BP′ = BR′ = AR′ − AB $= s - c$,

CP′ = CQ′ $= s - b$.

Ex. 1. Show that the mid point of BC bisects PP′.

Ex. 2. Write down the lengths of tangents from A, B, C to the ecircles in angles B, C.

Theorem 84.—'Two angles at the centre of a circle, their arcs, and the areas of their sectors are proportional.'



If AOB, COD are angles at the centre O of a circle,

make a scale of arcs, unit AB, from C along CD, divide decimally the unit containing D, join points of divisn. to O.

These joins form a similar scale of angles or sectors, unit AOB;

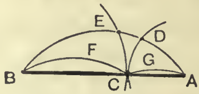
and D, OD come between the same divisions of the two scales;

$$\therefore \frac{\text{ang. COD}}{\text{ang. AOB}} = \frac{\text{arc CD}}{\text{arc AB}} = \frac{\text{sector COD}}{\text{sector AOB}}.$$

* s stands for $\frac{a+b+c}{2}$; i.e. semi-sum of sides.

Theorem 85.—'The straight line is the shortest path in a plane between two points in the plane.' *

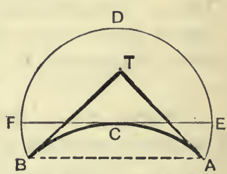
If ADB is a path from A to B not coinciding with str. line AB —e.g. not passing through C;
draw circles CD, CE, centres A, B.



Then part of the path, DE, is outside the circles; †
turn the parts AD, BE round A, B in the plane into the positions AGC, BFC;
∴ path AGCFB < ADB;
∴ not all paths are equal, and there must be some shortest path or paths;
and no path not coinciding with AB is the shortest;
∴ the str. line AB is the shortest path from A to B.

Cor.—Any arc of a curve is greater than its chord.

Theorem 86.—'The sum of the tangents from a point to a circle is greater than their enclosed arc.' (Legendre.)



If TA, TB are two tangents enclosing (with their chord) the arc ACB; then if ADB is any path enclosing ACB, and not coinciding with this arc,
draw tangt. ECF;
∴ path AECFB < ADB. (Th. 85.)
Hence no path not coinciding with ACB is the shortest enclosing path, and there must be some shortest;
∴ arc ACB < any enclosing path < sum of tangts. TA, TB.

Theorem 87.—'The circumference of a circle is greater than the perimeter of any inpolygon, and less than that of any circumpolygon.'

This follows at once from the previous theorems.

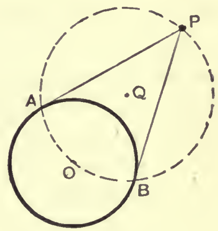
Ex. Show that circumfnce. of circle, rad. $r, > 6r$, but $< 8r$.

* Adapted from Francis Newman's *Difficulties of Elementary Geometry.*

† Because these are on opp. sides of the tangt. at C.

TANGENTS.

Construction 23.—'Draw two tangents to a circle from an outside point.'

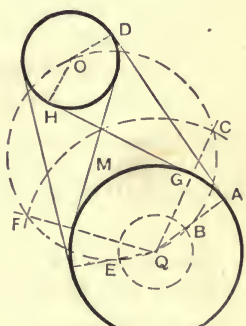


If P is the point, O centre of circle; bisect OP in Q, draw arcs of circle A, B, centre Q, diameter OP; join PA, PB.

Then PA, PB $\perp$ OA, OB respectively (ang. in semcle.);

$\therefore$ PA, PB are tangts. from P to circle.

Construction 24.—'Draw common tangents to two circles.'



If O, Q are centres of the circles DH, AG, with centre Q of the larger circle, draw two new circles BE, CF, radii the diffce. and sum of radii of given circles.

Bisect OQ at M, and draw circle, centre M, diameter OQ, cutting the new circles in B, C, E, F.

Join QB, QC to meet the circle AG in A, G;

make OD $\parallel$ QA, OH $\parallel$ QC;

then DA and HG are two common tangts.

For AB = and $\parallel$ OD ($\because$ QB = QA − OD);

$\therefore$ DA $\parallel$ OB $\perp$ QA and OD;

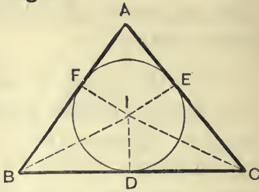
$\therefore$ DA, and similarly GH, is a common tangt.

In the same way two other common tangents are derived from the points E, F.

Ex. Discuss the cases (i.) when the circles touch, (ii.) when the circles cut.

Note. The common tangents are more simply drawn as tangents from the centres of similitude to either circle as a theoretical construction; but it is often inconvenient, as the outer centre of similitude may be outside the limits of the paper. They may be easily drawn by eye.

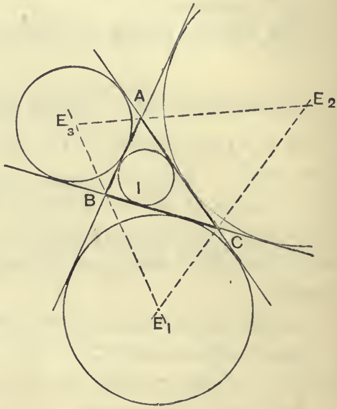
Construction 25.—'Draw a circle to touch three given straight lines; or
Draw the incircle and ecircles of a triangle.'



(i.) The incircle.

If the three lines form a tr. **ABC**, bisect angles **B, C** by **BI, CI**, draw **ID, IE, IF** perp. to **BC, CA, AB**; then **I** is equidistant from **BC, CA, AB**.

Draw circle, centre **I**, radius **ID**; this touches **BC, CA, AB** at **D, E, F**.



(ii.) The ecircles.

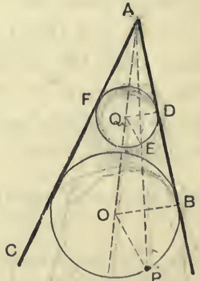
If the ext. angles at **B, C,** &c. are bisected by **BE_1, CE_1,** &c., circles with centres **E_1, E_2, E_3** can be drawn to touch **BC** and **AB, AC** produced, &c.

These are the three ecircles of the triangle **ABC**.

Thus there are four circles touching three straight lines.

Ex. Discuss the cases when (i.) two lines are parallel, (ii.) three lines are parallel, (iii.) three lines are concurrent.

Construction 26.—'Draw a circle to touch two given straight lines and pass through a given point.'



If **AB, AC** are the lines, **P** the point, draw **AO** bisecting ang. **A**; with any point **Q** in **AO** as centre, draw a circle to touch **AB, AC** in **D, F**, and to cut **AP** in **E**; draw **PO** parl. to **EQ**, and draw circle **PB**, centre **O**, touching the lines.

By similar triangles, if $OB \perp AB$,

$OP : QE = AO : AQ = OB : QD$;

$\therefore OB = OP$.

Ex. Show that there is a second circle.

Construction 27.—'Draw a circle to touch a given circle and two given straight lines.'

Draw parallels to the given lines at the distance of the given radius. The centre of a circle touching these parallels and passing through the centre of the given circle is centre of the required circle. (Constr. 26.)

Construction 28.—'Draw a circle through two given points to touch a given line.' *

If A, B are the points, CD the line, draw QD the rt. bisector of AB; from H, any point on it, as centre, draw circle GEF touching CD at G, and cutting AD at E, F.

Draw AO parl. to HE, and ON perp. to CD.

Then ON : HG = OD : HD = OA : HE;

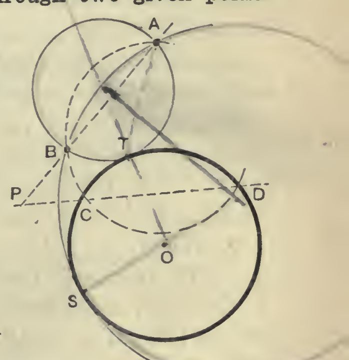
∴ ON = OA = OB.

Draw circle, centre O, rad. ON or OA touching CD at N, and passing through A, B.

If AQ || HF, Q is another centre.

Ex. Can the problem always be solved?

Construction 29.—'Draw a circle through two given points to touch a given circle.'



Draw a circle through given points A, B cutting the given circle, centre O, in C, D;

join CD to meet AB in P.

Construct S, T, points of contact of tangts. from P to given circle.

Draw circles ABT, ABS.

These touch the given circle at T, S.

For $PT^2 = PC \cdot PD = PA \cdot PB$;

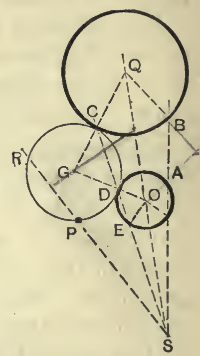
∴ PT is mean propl. of PA, PB, and touches circle ABT;

∴ circle ABT, and similarly ABS, touches given circle.

* This construction has been substituted for the traditional one because of its analogy with Constr. 26.

Construction 30.—'Construct a circle to touch two circles and pass through a given point.'

Construct a centre of simde. S of the given circles, centres O, Q, draw a secant SAB (A, B inverse points), find point R in SP such that rect. SP . SR = SA . SB.*



Draw circle, centre G, through R, P to touch circle QB in C. (Constr. 29.)

This also touches circle AD.

If SC cuts circle AD in E, D,
rect. SD . SC = SA . SB = SP . SR;
∴ D is on circle RPC;
and OE || QC (Th. 77);
∴ ang. GDC = GCD = OED (alt. ang.) = ODE;
∴ GDO is a str. line, and circle RPC touches ADE.†

Note. There is a second circle (Constr. 29) through RP touching circle BC, and therefore also touching AD; and two other circles through P touching the two circles, derived from the other centre of similitude.

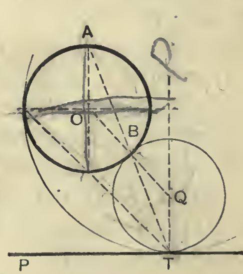
Construction 31.—'Construct a circle to touch three circles.'

If O, Q, R are centres of circles X, Y, Z in order of magnitude, diminish the radii of the greater X, Y by that of the least Z, draw circles X′, Y′, centres O, Q, with the diminished radii.

The centre P of a circle through R touching X′, Y′ is centre of the required circle.

Construction 32.—'Construct a circle to touch a given circle, and a given line at a given point.'

If T is the point in line PT, O the centre of given circle AB, make TQ perp. to PT. Make OA parl. to TQ, join AT to cut given circle in B, join OB to meet TQ in Q. Then
ang. QTB = alt. ang. OAB = OBA = QBT;
∴ QB = QT, and circle BT, centre Q, touches PT and circle AB.

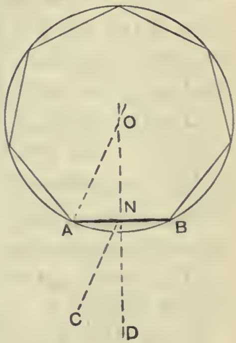


Ex. Find the centre of the other circle.

* SR is 4th propl. of SP, SA, SB.

† The points of contact C, D are centres of similitude of circles G, Q and G, O, and are collinear with a centre of simde. of O, Q.

Construction 33.—'Construct a regular polygon of n sides on a given side.'



(Same construction as arc to contain given angle.)

If AB is the side, draw the rt. bisector DNO; and make ang. DNC = the nth part of *two* rt. angs.; *
draw AO parl. to CN,
draw circle AB, centre O;
∴ ang. AOB = 2 . AON = 2 . CND
= nth part of 4 rt. angs.;
∴ arc AB = nth part of circle.

Step off the arc AB round the circle and complete polygon as in figure.

Note. (i.) For a hexagon, make AOB an eql. triangle.

(ii.) For a pentagon,† construct an isosceles triangle AFB whose base AB is the mean part of sides FA, FB.

With A, B, F centres, radius AB, draw arcs meeting in G, H; AGFHB is the pentagon.

Construction 34.—(i.) 'Inscribe a regular polygon in a given circle.'

Take any radius OA of the circle;
make ang. AOB = the nth part of *four* rt. angs.;
∴ arc AB = nth part of circle; complete as before.

Note. (i.) For a hexagon, chd. AB = AO.

(ii.) For a pentagon,† construct on OA an isosceles triangle FOA of which OA is the mean part of sides OF, FA;
FO meets the circle in B. Complete as before.

(ii.) 'Circumscribe a regular polygon to a given circle.'

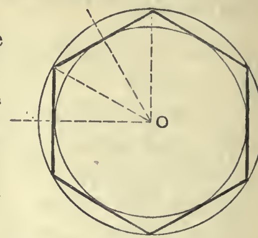
Divide the circle into n parts as in last constr.
Draw tangents at the points of division.

Ex. Inscribe and circumscribe a regular octagon to a circle.

* In the example $n=7$, DNC $=180°/7=26°$ nearly.

† If the exact geometrical construction is required.

Construction 35.—'Draw the circumcircle and incircle of a given regular polygon.'

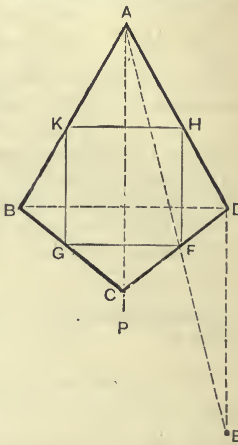


Draw rt. bisectors of two consecutive sides of the polygon, meeting in **O**; or draw bisectors of two consecutive angles, meeting in the same point **O**.

O is the centre of both circles.

This is easily made evident by rotation of the figure about **O**.

Construction 36.—'Inscribe a square in any figure which is symmetrical about a bisector of angle.'



(This includes a sector of circle, and any regular polygon.)

If fig. **ABCD** is symmetrical about **AP**, the bisector of ang. **A**, so that **AB** = **AD**; set off **DE** perp. and eql. to **BD**.

Join **AE** to meet the figure in **F**; draw parls. to **BD**, **DE**, starting from **F**, forming the square **FGKH**.

For $\frac{KH}{BD} = \frac{AH}{AD} = \frac{HF}{DE}$, and $BD = DE$;

∴ $KH = HF$;

∴ **FGKH** is an equal-sided rectangle—i.e. a square.

MULTIPLICATION.*

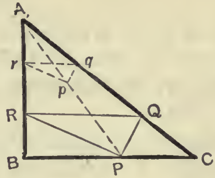
This construction and Constrs. 26 and 28 are examples of the general method of multiplication. We construct a figure similarly situated to that required, satisfying all the required conditions but one; we then enlarge or reduce—i.e. multiply—to obtain the figure satisfying this remaining condition (Ch. III. **A** (iii.), p. 83).

Thus, above, **BD**, **DE** are sides of a square satisfying all conditions except having a vertex on **CD**, so that the required vertex of the similarly situated square lies on **AE**; we thus obtain **F**. Similarly in Constrs. 26, 28, the circles **DEF**, **EFG** satisfy all the conditions except passing through **P**, **A** respectively.

A few examples of the method are given on the next page.

* A very good account of this and other general methods of construction is given in Pedersen's *Méthodes et Théories*.

Ex. (i.). 'Inscribe in a given triangle ABC a triangle whose sides have given directions.'



Make rq in one of the given directions, and rp, qp in the other two directions.

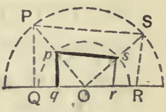
Join Ap to P on BC;

and make PR, PQ parl. to pr, pq.

Then AQ : Aq = AP : Ap = AR : Ar;

∴ QR ∥ qr, and PQR is reqd. triangle.

Ex. (ii.). 'Inscribe in a semicircle (with two vertices on the diameter) a quadrilateral of given form (similar to a given quadrilateral $pqrs$).'

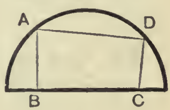


Draw rt. bisector of ps to O in qr; then $pqrs$ is inscribed in a semcle., centre O.

Draw semcle. PS, centre O, diam. along qr, eql. to given semcle.

From O multiply $pqrs$ into PQRS;

copy PQRS into ABCD inscribed in the given semicircle AD.



EXAMPLES—XXV.

1. Inscribe in a triangle ABC an isosceles right triangle having its hypotenuse parallel to BC.

2. Inscribe a square in a given segment of a circle. (Two vertices on the base of the segment.)

3. Place a chord in a circle so as to be trisected by two radii whose angle is 30°.

4. Place a chord in a circle in a given direction, and divided in a given ratio by a fixed diameter.

5. Inscribe a triangle PQR in a given triangle ABC, having given P in position on BC, the angle P, and direction of QR.

6. Draw a straight line through a given point P to cut two given circles at A, B, so that PA : PB = ratio of corresponding radii.

7. Inscribe a parallelogram, sides 2 : 1, angle 60°, in a semicircle.

8. Draw a straight line in a given direction to divide the opposite sides AB, DC of a quadrilateral in the same ratio.

9. Inscribe a square in a triangle. (Two vertices of the square on one side of the triangle.)

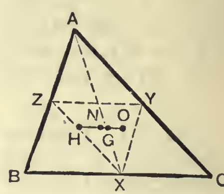
10. Inscribe a square in a sector of a circle.

11. Inscribe a parallelogram in a segment of a circle, chd. 1″, rad. $\frac{3}{4}$″ (two vertices on base), one angle 50°, ratio of sides 2 : 3.

We add an example of the method as applied to the proof of theorems.

Ex. (iii.). 'The centroid, circumcentre, orthocentre, and nine-point centre of a triangle are collinear.'

If G, O, H, N are the above points in order in a triangle ABC; the median triangle XYZ can be derived from ABC by multiplication from G by the ratio GX : AG = 1 : 2.



Hence any point of tr. XYZ is collinear through G with the corresponding point of ABC.

But O, the circumcentre of ABC, is the orthocentre of XYZ,

∵ XO, YO ⊥ ZY, ZX;

∴ corresponding orthocentres H, O are collinear with G.

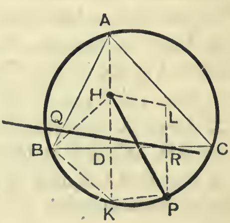
Similarly O, N, the circumcentres of ABC, XYZ, are collinear with G; i.e. G, O, H, N are collinear.

Ex. Find the ratios of parts of HO.

The following example does not illustrate multiplication, but is an interesting property of the triangle, which we require for a property of the parabola in Ch. VIII.

Ex. (iv.). 'The Simson line of a point on the circumcircle of a triangle bisects the join of the point to the orthocentre of the triangle.' *

If QR is Simson line of point P on cmcirc. of tr. ABC, and AHDK the perp. to BC through orthocentre H;
make RL eql. to PR in line PR.



Then ang. KBC = KAC, same arc,
= compt. of C = HBD;

∴ KD = DH; also PR = RL;

∴ trapm. LHKP is symml. about RD;

∴ ang. HLR = RPK = suppt. of PKH
(∵ PR ∥ KD)
= suppt. of PBQ, same arc,
= QRP, opp. arc;

∴ QR ∥ HL, and bisects HP.

* Casey's *Sequel to Euclid*, Bk. III.

EXAMPLES—XXVI.

THEOREMS.

1. The locus of mid points of a system of parallel chords of a circle is a diameter of the circle.

2. The joins of the ends of two parallel chords of a circle meet in **P, Q.** Show that **PQ** passes through the centre. (Use symmetry.)

3. Perpendiculars on a chord of a circle from the ends of a diameter cut off equal lengths from the line of the chord, measured from its two ends.

4. The centre of any circle is the centre of the circle which touches any three equal chords.

5. Straight lines from a point to a circle increase in length as the angles they subtend at the centre increase from zero to two right angles.

6. Two circles do not meet at all if the join of their centres is greater than the sum or less than the difference of their radii.

7. When do circles (i.) cut, (ii.) touch, (iii.) not meet at all?

8. The sum of two opposite sides of a quadrilateral circumscribing a circle is equal to the sum of the other sides.

9. A parallelogram circumscribing a circle is a rhombus.

10. A tangent **PQ** of a circle, centre **O**, cuts two fixed tangents in **PQ**; show that the angle **POQ** is constant. What is its value when the fixed tangents are parallel?

11. The chord of contact of two tangents forms with them an isosceles triangle.

12. The chord of contact of two parallel tangents is a diameter.

13. Each side of a rhombus circumscribing a circle subtends a right angle at the centre.

14. The bisector of an angle in an arc of a circle bisects the opposite arc.

15. If **AP, AQ** are diameters of two circles cutting in **A, B**, show that **BP, BQ** are in a straight line.

16. The arcs between two parallel chords of a circle are equal.

17. A parallelogram inscribed in a circle is a rectangle.

18. The joins of ends of two equal arcs of a circle are either equal or parallel.

19. The joins towards the same parts of the ends of two equal chords of a circle complete a trapezium.

20. A circle cuts off equal chords from the two sides **PA, PB** of an angle **P.** Show that the lengths **PA, PB** outside the circle are equal.

21. Chords BAC, DAE drawn through a common point A of two circles, and equally inclined to the line of centres, are equal.

22. A triangle is equilateral whose incentre and circumcentre coincide.

23. Two arcs stand on a common chord AB. AP, BP meeting on one arc cut the other in Q, R. Show that the chord QR has constant length.

24. The sides AB, DC of a quadrilateral in a circle meet at P, and AD, BC at Q; show that the bisectors of angles P, Q are perpendicular.

25. If two triangles have a side and opposite angle of one equal respectively to a side and opposite angle of the other, their circumradii are equal.

26. If DE parallel to the unequal side BC of an isosceles triangle ABC cuts the sides in D, E, the points D, E, C, B are concyclic.

27. If the ends of two chords AB, CD of a circle are cross joined, the two triangles formed are similar.

28. The bisector of the angle PTA of a tangent PT and chord TA also bisects the arc TA.

29. If the bisector of the angle ACB in an arc of a circle is parallel to the tangent AP meeting BC produced in P, then AP = AB.

30. If two circles, centres O, Q, touch at T, and ATB is any chord through T, then AO, BQ are parallel, and arcs AT, BT are similar.*

31. If A, B, C are points in order in a straight line, similar arcs on AB, AC, on the same side of the line, touch at A; and similar arcs on AB, BC, on opposite sides, touch at B.

32. If a circle touches a straight line AB at P, and a circle through AB touches the first circle in Q, the angles AQP, BQP are either equal or supplementary.

33. The radius of the nine-point circle of a triangle is half that of the circumcircle.

34. The internal common tangent of two circles which touch externally bisects the other common tangents.

35. If PT, PS are tangents to a circle, centre O, and OT meets PS in Q, show that QP . QS = QT . QO.

36. If a straight line cuts two intersecting circles in A, B and C, D, and their common chord in P, then PA . PB = PC . PD.

37. The tangents to two intersecting circles from a point on their common chord are equal.

38. Enunciate Ex. 37 when two circles touch.

39. The line of any chord AP through a fixed point A on a circle cuts in Q a fixed line parallel to the tangent at A. Show that AP . AQ is constant.

40. A tangent PT of a circle at T is equal to the radius, and TA is the diameter through T; show that the circle cuts off four-fifths of PA.

* That is, their angles are equal.

41. CD is the bisector of angle C of a triangle whose angles B, C are each double of A. Show that BC touches the circumcircle of ADC.

42. If I, E are incentre and ecentre on bisector AP of angle A of a triangle, AIPE is harmonically divided.

43. If I is the incentre of a triangle, and AI meets BC in P, show that PI : PA $= a : a+b+c$.

44. The point of contact of two circles which touch is a centre of similitude of the circles.

45. If AP is a bisector of angle of a triangle, A, P are centres of similitude of the incircle and an ecircle.

46. The centroid of a triangle is a centre of similitude of the circumcircle and nine-point circle.

47. A straight line through a vertex A of a triangle cuts the incircle and ecircle in inverse points P, Q; show that AP . AQ $= s(s-a)$. (A is a centre of simde.)

48. A tangent TA at T of a circle, centre O, is equal to a diameter and is bisected at M. Show that MO is equal to the tangent from A to circle MTO.

49. If AD, BE are perpendiculars of a triangle, orthocentre H, show that CA . CE = CB . CD; also that DH . DA = DB . DC.

50. If two circles cut at right angles, the square of their join of centres is the sum of squares of their radii.

51. If two circles are each touched by a third, the join of points of contact passes through a centre of similitude of the two circles.

Can you connect this fact with the construction for drawing a circle to touch two given circles?

52. Two circles, centres O, Q, cut in A, B. OC, QD are parallel radii of their respective circles in opposite directions, and CD meets OQ in S. Show that

(i.) S is a centre of similitude;

(ii.) the orthocentres of triangles SOC, SQD are collinear with S.

53. Show that in Ex. 52, S must be inside each circle, and hence that two circles which cut in two points cannot have any internal common tangents.

54. If the points of contact of the external common tangents of two circles are collinear, two and two, with the internal centre of similitude, show that the circles are equal.

What is the internal centre of similitude of two circles which touch?

EXAMPLES—XXVII.

CONSTRUCTIONS.

1. Draw a circle, with given centre, to cut off a given length from a given line.

2. Construct the locus of mid points of equal chords of a circle.

3. Draw a straight line through a fixed point so that a fixed circle cuts off a given length from it.

4. Draw a circle, 2·2 cm. radius; take a point P, 2·8 cm. from the centre; draw a circle, centre P, to pass through points of contact of tangents from P to the first circle.

5. Draw a circle, 1·35″ radius; take a point 1·08″ from the centre, and construct the shortest chord through this point. Measure the chord.

6. Through a point A common to two circles draw a chord BAC bisected at A. Can this be done if the circles touch?

7. Draw a circle, radius 1·65″, place in it a chord of 1·32″ parallel to a fixed direction.

8. Find the longest chord BAC through the common point A of two circles.

9. Construct a square of 1·47″ side, and circumscribe a circle.

10. Draw a circle, radius 1·73″; cut off an arc whose chord is 1″ and bisect the arc.

11. Draw a circle, radius $\frac{3}{4}$″, to touch two straight lines forming an angle of 40°.

12. Draw a circle through the centre and two fixed points of a given circle.

13. Inscribe a circle in a square of 2·37″ side.

14. Inscribe a square in a circle of 5·6 cm. diameter, and measure the side.

15. Construct a rectangle, sides 2·4 cm. and 3·2 cm.; circumscribe a circle, and measure the radius.

16. A ladder 30 ft. long is placed horizontally, perpendicular to and just touching a wall. It is then raised with one end always touching the wall. Construct the locus of its mid point (1″ to 10 ft.).

17. Construct a point in a fixed line such that two fixed points off the line subtend at it a given angle. How many solutions?

18. Draw a circle, radius 2·7 cm., and a chord 3·2 cm. Find a point on the circumference equidistant from the ends of the chord.

19. Construct the locus of a point such that the sum of its greatest and least distances from a fixed circle is constant.

20. Inscribe a circle in a sector of a circle, angle 60°, radius 2″. Measure the radius of the incircle.

21. Inscribe a regular heptagon in a circle, radius 2 cm.

22. Describe a regular hexagon about a circle, radius 1·8 cm.

23. Construct the locus of points whose tangents to a given circle have a given length.

24. Draw two tangents to a circle to contain a given angle.

25. Draw a rhombus, angle 87°, to circumscribe a circle.

26. Draw a circle through the mid point of one side of a triangle to touch the other two sides. How if the triangle is isosceles?

27. Draw a semicircle, radius 2·3 cm. Inscribe in it a circle, radius 1 cm.

28. On two sides of an angle A take equal lengths BC, PQ, $1\frac{1}{4}''$ long, $\frac{1}{2}''$ and $\frac{3}{4}''$ from A respectively. Verify that the circles APC, ABQ and the right bisectors of CP, BQ are concurrent.

29. Construct a regular pentagon, not using the protractor, on a side of 1 inch. Inscribe a circle.

30. Find in a given straight line a point whose tangent to a given circle has a given length.

31. Given any two points A, B, construct the length of the tangent from a point P in AB produced to any circle through A, B.

32. Draw a circle passing through the mid point of one side of a triangle, the foot of the perpendicular of another side, and touching the third side.

33. Find a point P in a given line AB such that the rectangle $\mathsf{PA}\,.\,\mathsf{PB}=a^2$, where a is a given length.

34. Construct the locus of a point whose distances from two fixed points $1\frac{1}{2}''$ apart are in the ratio 3 : 2.

35. Draw a circle, radius 2·5 cm., cut off an arc whose chord is 3·1 cm., and divide the arc into two parts whose chords have the ratio 3 : 4.

36. Draw an isosceles triangle, angle of equal sides 50°, third side $1\frac{1}{4}''$. Draw two equal circles each to touch the other, one of the equal sides, and the third side of the triangle.

37. Inscribe in an equilateral triangle three equal circles to touch each other.

38. Construct three equal circles to touch each other and to touch internally a circle of 2·8 cm. radius.

39. Draw three circles, with the points A, B, C of a triangle as centres, to touch each other.

40. Draw a sector of radius $1\frac{3}{4}''$, angle 120°. Inscribe a square in it.

41. Inscribe in a square of 1·63″ side a parallelogram of angle 78°, having one vertex $\frac{1}{2}''$ from a vertex of the square.

42. Construct a cyclic quadrilateral, given diagonals **AD** and **BC**, the angle **A**, and the ratio **AB : AC**.

43. Construct a quadrilateral, given diagonals $\mathbf{AC}=1\frac{1}{2}''$, $\mathbf{BD}=1''$, their angle 108°; $\mathbf{A}=60°$, $\mathbf{C}=71°$. (Translate tr. **BAD** from **A** to **C**.)

44. Construct a quadrilateral, given the sides **AB, BC, CD** and the angles **CAD, CBD**.

45. Circumscribe a rhombus of 7·2 cm. side about a circle of 2·7 cm. radius.

46. Construct a point **P** in the side **BC** of a triangle such that the square on **PA** is equal to the rectangle **PB . PC**.

47. If **P** is any point in the side **BC** produced of a triangle, construct a transversal **PDE** such that rect. **PD . PE = PB . PC**.

48. Describe an octagon on a side of 2·3 cm. Give a general construction for an inscribed square with a vertex at any point on a side of the octagon.

49. Construct a regular pentagon, diagonal 5·7 cm. (The side is the mean part of the diagonal.)

50. Given the angle **A** of a triangle, construct the locus of the mid point of **BA** when vertices **B, C** are fixed.

51. The parallel sides **BA, CD** of a trapezium are given in magnitude, and **B, C** are fixed points. Find the locus of intersection of bisectors of angles **B, C**, also of the mid point of **AD**.

52. Draw the nine-point circle of a triangle whose sides are 4 cm., 5 cm., 6 cm. Test it at the feet of perpendiculars.

53. Draw a circle, radius $1''$. Inscribe a triangle, two sides $1\frac{1}{2}''$, $1\frac{3}{4}''$. Construct its orthocentre **H**; measure the parts into which the Simson line to the triangle of any point **P** on the circle divides **PH**.

54. Construct the absolutely shortest path from a given point to a given circle. Indicate the proof. Assuming that maxima and minima occur alternately, what is the longest path?

55. Draw a circle to touch two lines at an angle of 60°, and a circle of $\frac{1}{2}''$ radius, whose centre is distant $\frac{3}{4}''$ and $1\frac{1}{4}''$ from these lines.

56. Draw a triangle, $a=4$ cm., $b=6$ cm., $c=7$ cm. With centre **C**, rad. 3 cm., draw a circle. Draw a circle through **A, B** to touch this circle.

57. Construct a mean base triangle (Constr. 16, Ch. III.), and in it inscribe an isosceles right triangle, whose hypotenuse is parallel to an equal side of the first triangle.

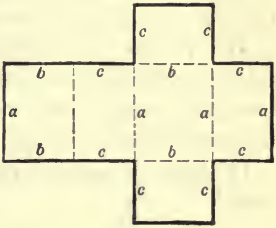
58. Draw a sector whose angle is 60°, radius $1''$. Inscribe in the sector a triangle whose sides are as 2 : 3 : 4, the greatest side parallel to chord of sector.

59. Inscribe in a semicircle, 3 cm. radius, a parallelogram of angle 70° and sides 1 : 3 (two vertices on arc, two on diam.).

Solid Figures—Volume—Cube—Cuboid.

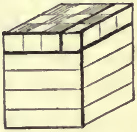
Figures whose points, lines, surfaces are not all in one plane are **solid** figures. When closed they include a certain amount of space called their volume or cubic content, the unit being the cube whose edge is unit length. (Cubic inch, &c.)

Draw and cut out the adjoining figure of rectangles (thick paper or thin board).* Fold along the dotted lines to form the second figure; fasten the edges with gummed paper. This makes a rectangular solid or cuboid.



The lines a, b, c, sides of the rectangles, are called the **edges**, and the rectangles the **faces** of the cuboid.

Make a second figure, in squares instead of rectangles. This makes a cube, and a side of a square is the edge of the cube. Thus,



A **cube** is a solid formed by six equal squares.

A **cuboid** is a solid formed by six rectangles.

Volume of a Cuboid.

If the edges of the cuboid are divided into unit lengths, 5, 4, 3 units, say;
then we can make 5 cuboids, edges 1, 4, 3;
and each of the last makes 4 cuboids, edges 1, 1, 3;
and each of the last makes 3 cuboids, edges 1, 1, 1—i.e. 3 unit cubes.

Thus we have $5 \times 4 \times 3$ unit cubes. Hence,

'The volume of a cuboid is the product of its three edges, or of its height, length, and breadth.' †

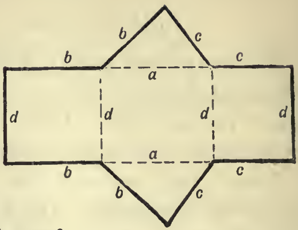
Ex. 1. Find the volume of a box 30″ by 18″ by 15″; and of a room 25 ft. by 20 by 12.

Ex. 2. Find the volumes of cubes of edge 7″, 13 ft., 2·4 cm.

* A post-card serves well.

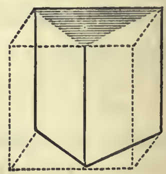
† The complete proof is given below, Ch. IX.

Wedge.



Make and cut out the adjoining figure of three rectangles and two triangles, whose sides are those of the rects., fold along dotted lines, and fasten the edges, making a wedge.

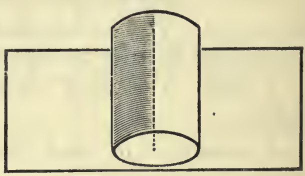
A (right) **wedge** is a solid formed by three rectangles and two triangles.



The volume of a wedge is half that of a cuboid whose rectangular base is twice the triangular base of the wedge, and which has the same altitude; thus,

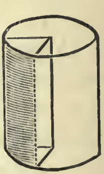
'Volume of wedge = triangular base × altitude.'

Cylinder.



Cut out a rectangle and fold it round* so that two opposite sides coincide and the others form equal circles.

This with the circular ends is a (right) **cylinder.** The join of the centres of the ends is its **axis,** and either end is the **base.**



Area of a right cylinder, radius of base r, altitude h.

The area of each base is πr^2.

Area of curved surface is the rectangle, altitude × length of base circle ($h \times 2\pi r$).

'Area of curved surface of cylinder = height × circumference of base circle.'

Volume of a Cylinder.

If planes through the axis cut out small sectors from the ends, they cut out practically wedges from the cylinder, whose volume is height × area of sector.† Thus,

'The volume of a cylinder = height × area of base.'†

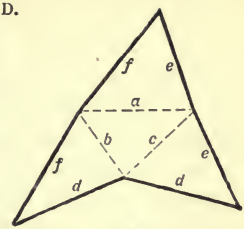
Ex. 1. Calculate volume of wedge, alt. 2·3 cm., base area 5 sq. cm.

Ex. 2. Calculate area and volume of cylinder, alt. 6 cm., rad. 1·2 cm.

* It can be first folded round a bar or tube, and cut as a rectangle afterwards.

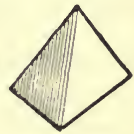
† This is strictly true, though the proof is not complete.

TETRAHEDRON—PYRAMID.



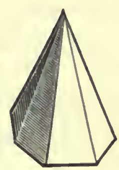
Cut out the adjoining figure of four triangles, adjacent sides of any two outer triangles being equal, fold along dotted lines, and fasten the edges, making a tetrahedron.

A **tetrahedron** is a solid formed by four triangles.



If instead of the triangle *abc* we take a polygon, and make triangles on its sides as before, the solid figure is a pyramid.

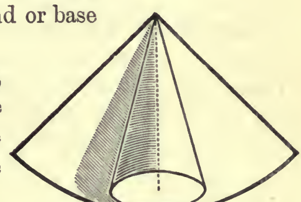
A **pyramid** is a solid formed by a polygon and triangles based on its sides, and having a common vertex.



'The volume of a tetrahedron or pyramid is a third of the product, height × base.' *

CONE.

Cut out a sector of a circle, and fold round so that the sides coincide, and the arc forms a circle. This with the plane circular end or base is a (right) **cone.**



The centre of the sector is the **vertex,** the join of vertex to centre of base the **axis** and **altitude** or height; and the join of vertex to any point on the base circle is the **slant height** of the cone.

The area of the curved surface of the cone is that of the sector which forms it. Thus,

'The area of curved surface of a cone is half the product, slant height × length of base circle.'

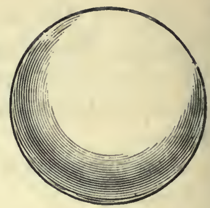
'The volume of a cone (like that of a pyramid) is a third of product, height × base.' *

Ex. 1. Calculate the volume of a pyramid, height 5″, base a square of 2·7″ side.

Ex. 2. Calculate surface and volume of a cone, height 4 cm., slant height 5 cm., radius of base 3 cm. How could you calculate the slant height?

* The proof is given below, Ch. IX.

SPHERE.



A **sphere** is a closed surface whose points are all equidistant from a fixed point, its centre. The distance from centre to surface is the **radius,** and from surface to surface through the centre the **diameter** of the sphere. A billiard ball is a good example.

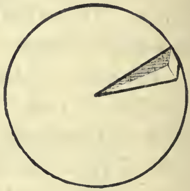
Any plane cutting the sphere cuts it in a circle—e.g. a line of latitude on a globe of the world.

Any plane through the centre cuts the sphere in a **great circle** —e.g. line of longitude.

Great circles can be moved about on a sphere just as straight lines on a plane; and one only can be drawn between two points, unless these are opposite ends of a diameter of the sphere.

'The area of a sphere is four times the area of a great circle;'* i.e. area = $4\pi r^2$.

VOLUME OF A SPHERE.



If the sphere is divided into small pyramids by planes through the centre, the volume of a pyramid is $\frac{1}{3}$ area of base × altitude (radius). And the volume of the sphere—i.e. the sum of the pyramids—is that of a pyramid whose base is the surface of the sphere, and altitude the radius; thus—

'The volume of a sphere is one-third the product of radius and surface;' i.e. vol. of sphere = $\frac{4\pi r^3}{3}$.

Ex. 1. Calculate the surface and volume of spheres of radii 2 cm., 1·6″, 5 ft.

Ex. 2. Taking a degree as 69·12 miles, calculate the earth's circumference (length of a great circle).

Ex. 3. Taking the earth's radius as 3960 miles, calculate its circumference, and its area.

* The proof is given below, Ch. IX.

EXAMPLES—XXVIII.

1. Calculate the volume of a room 24 ft. long by 18 ft. wide and 12 ft. high.

2. Calculate the area of a box 33″ by 16″ by 18″.

3. Calculate the volume of a pile of books 100 deep, each book $8\frac{1}{2}''$ by 6″ by $1\frac{1}{4}''$.

4. How much water is required to fill a stretch of canal 3 miles long, 6 ft. deep, 24 ft. wide? Answer in cubic yards.

5. A wedge has a triangular base of sides 3″, 4″, 5″, and an altitude of 6″. Calculate its volume.

6. A cuboid block of wood 2 ft. by 1 ft. by 3″ is sawn across the diagonals of its greatest faces so as to form four wedges. Calculate the volume of each.

7. How many cubic inches of milk does a can 1 ft. high, 1 ft. diameter, hold?

8. If 1 gallon contains 277 cubic inches, how many gallons does the can of No. 7 hold?

9. A bullet tied to one end of a string is swung round in a circle, the other end of the string being fixed; what surface does the string generate? If the diameter of the circle is 10″, and length of string 21″, what is the area of the curved surface?

10. Taking the height of the Great Pyramid as 480 ft., and the side of the square base as 762 ft., calculate the volume in cubic yards.

11. A conical funnel has a diameter of 4″ and a depth of 6″. What volume of water will just fill it?

12. A conical church spire is 16 yd. high, on a base of 8 yd. diameter. Calculate its volume, and the area of its curved surface.

13. A sphere can just be got into a cylindrical vessel 2 ft. high, 2 ft. diameter. Calculate the volume of the sphere.

14. What volume of water would just fill the cylinder in Ex. 13 with the sphere, supposed solid, inside?

15. The area of the curved surface of a cylinder is 94·2 sq. in. If the height is 10″, what is the diameter of the base?

16. How many gallons of water does a cask hold, 6 ft. long, average diameter 3 ft.? (1 cub. ft. = 6·25 gallons.)

17. Show that 1″ of rainfall is equivalent to 101 tons per acre. (1 cub. ft. of water weighs 1000 oz.)

18. Calculate the area of a triangular face of the Great Pyramid of Ex. 10.

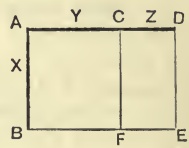
CHAPTER V.

RECTANGLES BY ALGEBRAIC FORM—MEASURE AND RATIO OF AREAS—RECTANGLE PROPERTIES OF THE TRIANGLE—MISCELLANEOUS.

Rectangles by Algebraic Form.

The area of a rectangle being determined by its sides X, Y may be written XY as an algebraic product. The area of the square on side X may be similarly written XX or X^2.

Theorem 88.—(i.) 'The rectangle whose sides are a line X, and the sum of two lines $Y + Z$, is equal to the sum of rectangles $XY + XZ$.'

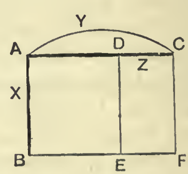


If AB is X, and AC, CD in a perp. line are Y, Z, complete rects. AE, AF.

Then $X(Y+Z) = \text{rect. } AE = AF + CE$

$= XY + XZ$ ....(i.).

(ii.) 'The rectangle whose sides are a line X, and the difference of two lines $Y - Z$, is equal to the difference of rectangles $XY - XZ$.'



If AB is X, and AC, CD in a perp. line are Y, Z, complete rects. AE, AF.

Then $X(Y-Z) = \text{rect. } AE = AF - CE$

$= XY - XZ$...........(ii.).

We thus derive a very important principle:

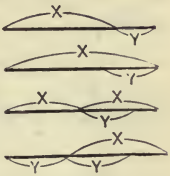
'Rectangles whose sides are the sum or difference of lines can be treated by the rules of algebraic products.'

For the two forms (i.) and (ii.) are the fundamental forms of products $x(y+z)$ and $x(y-z)$ in algebra.

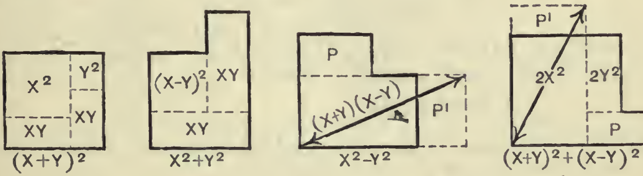
A few examples of the method are given. With due care as to sign ($+$ and $-$) results are readily obtained.

Theorem 89.—'If X and Y are any two straight lines'—

(i.) $(X+Y)^2 = X^2 + Y^2 + 2XY$;
(ii.) $(X-Y)^2 = X^2 + Y^2 - 2XY$;
(iii.) $X^2 - Y^2 = (X+Y)(X-Y)$;
(iv.) $(X+Y)^2 + (X-Y)^2 = 2X^2 + 2Y^2$.



These follow exactly as in algebra. The following diagrams illustrate them geometrically.



In the third and fourth figures the rectangle P must be moved into the position of the congruent rectangle P′.

Theorem 90.—'If a straight line AB is bisected at M, and P is any other point in the line, $PA^2 - PB^2 = 2 . MP . AB$.' $(PA > PB.)$

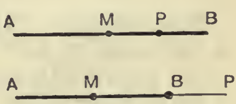
By Th. 89 (iii.) $PA^2 - PB^2 = (AP + PB)(AP - PB)$.

But when P is between A and B,

$(AP + PB) = AB$,

$(AP - PB) = MP + MB - PB = 2MP$;

$\therefore PA^2 - PB^2 = 2MP . AB$.



Similarly, when P is outside A and B,

$(AP + BP) = 2MP$, $AP - BP = AB$;

$\therefore PA^2 - PB^2 = 2MP . AB$.

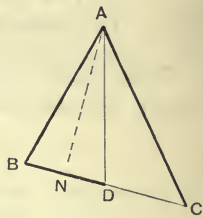
Cor.—'There is one only point P in a straight line the difference of whose squares of distances from two fixed points in the line in a given order has a given area.'

For if the given area $S = \text{rect. } X . AB$, then X is a fixed length; and $2MP . AB = PA^2 - PB^2 = S = X . AB$;

$\therefore MP = X/2$, a fixed length, and P is a fixed point.

Theorem 91.—'The sum of squares on two sides of a triangle is twice the sum of squares on half the third side and its median.'

If AD is the median of BC, in triangle ABC, make AN perp. to BC;



then $AB^2 + AC^2 = BN^2 + AN^2 + CN^2 + AN^2$

$= BN^2 + CN^2 + 2AN^2$;

but $BN^2 + CN^2 = (BD - ND)^2 + (DC + ND)^2$

$= (BD - ND)^2 + (BD + ND)^2$,

$\because BD = DC$,

$= 2BD^2 + 2ND^2$ (Th. 89, iv.);

$\therefore AB^2 + AC^2 = 2BD^2 + 2ND^2 + 2AN^2$

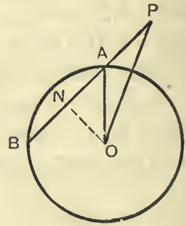
$= 2BD^2 + 2AD^2$.

Ex. Prove the theorem when the angle B is obtuse. What does the theorem become when B is a right angle?

Theorem 92.—'The square on one side of a triangle is greater or less than the sum of squares on the other two sides by twice the rectangle of either of these sides and the projection on it of the other, according as the angle opposite the first side is obtuse or acute.'

This can be proved in a similar manner to the last theorem. It is interesting only as completing Pythagoras' theorem.

Theorem 93.—'The rectangle PA . PB of the parts of any chord or secant of a circle, centre O, is equal to the difference of squares $OP^2 - OA^2$.'



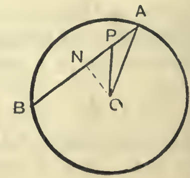
Draw NO, the rt. bisector of AB; then, if P is external,

$PA . PB = (NP - NA)(NP + BN)$

$= (NP - NA)(NP + NA)$

$= NP^2 - NA^2$ (Th. 89, iii.)

$= OP^2 - OA^2$ (by Pythagoras' theorem).

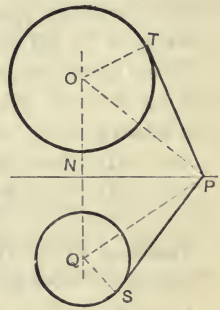


Similarly, if P is internal,
$AP . PB = OA^2 - OP^2$.

Definition 35.—The **radical axis** of two circles is the locus of points whose tangents to the circle are equal.

A **coaxial system** of circles is one of which all pairs have the same radical axis.

Theorem 94.—'The radical axis of two circles is a straight line perpendicular to the line of centres.'



If TP, SP are equal tangents to two circles, centres O, Q, and TO $\gtreqless$ QS, then P is a point on the radical axis.

Make PN perp. to OQ. Then

$$\begin{aligned} NO^2 - NQ^2 &= (PO^2 - PN^2) - (PQ^2 - PN^2) \\ &= PO^2 - PQ^2 \\ &= (PT^2 + OT^2) - (PS^2 + QS^2) \\ &= OT^2 - QS^2 = \text{constant}; \end{aligned}$$

∴ (Th. 90, Cor.) N is a fixed point, and NP a fixed line.

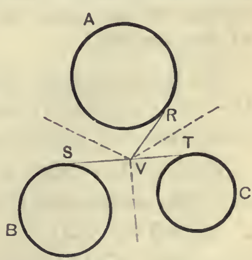
It is readily seen that every point outside the circles on the line NP has equal tangents to the two circles.*

Note. (i.) If the circles cut in A, B, the line AB is the radical axis.

For if RK, RL are tangents from any point R in AB,
$RK^2 = RA \cdot RB = RL^2$; ∴ $RK = RL$.

Note. (ii.) If two circles touch, the radical axis is the tangent at their point of contact.

Theorem 95.—'The three radical axes of three circles meet in a point.' (The **radical centre** of the circles.)



If the radical axes of circles A, B and A, C meet in V, and VR, VS, VT are tangents to A, B, C;

then VT = VR = VS;

∴ V is on the radical axis of B, C;

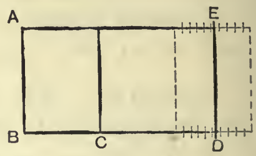
i.e. the three radical axes are concurrent.

Note. If V is inside the circles, prove by the rectangle property of chords.

Cor.—'If three circles touch each other, the common tangents at their points of contact are concurrent.'

* If a point on the line is inside one circle, it is also inside the other.

Theorem 96.—'**The areas of two rectangles of equal altitudes are proportional to their bases.**'

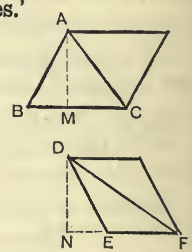


If rects. ABC, ABD are placed with their eql. alts. on AB, and their bases BC, BD along BD;
make a scale, unit BC, along BD,
divide decimally the unit containing D,
draw parls. to AB through points of division.

These form a scale of rectangles, unit AC;
and point D and side DE come between the same divisions of the two scales;

$$\therefore \frac{\text{rect. AD}}{\text{rect. AC}} = \frac{\text{base BD}}{\text{base BC}}.$$

Theorem 97.—'**The areas of two triangles or parallelograms of equal altitudes are proportional to their bases.**'

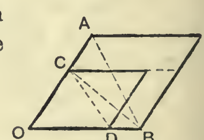


If ABC, DEF are triangles or parms. of eql. alts. AM, DN;

$$\text{then } \frac{\triangle\,\text{ABC}}{\triangle\,\text{DEF}} = \frac{\frac{1}{2}\text{ rect. AM . BC}}{\frac{1}{2}\text{ rect. DN . EF}} = \frac{\text{base BC}}{\text{base EF}};$$

$$\text{and } \frac{\square\,\text{AC}}{\square\,\text{DF}} = \frac{\text{rect. AM . BC}}{\text{rect. DN . EF}} = \frac{\text{base BC}}{\text{base EF}}.$$

Theorem 98.—'**The ratio of two rectangles, or of two parallelograms or triangles of given angle, is the product of ratios of the sides of the given angle of the first to those of the second.**'



Set the two rects., parms., or trs. AB, CD with the given angle at O, and the sides of this angle along OA, OB, and complete as in fig.

Then if $\text{OA} = \mu\text{OC}$, and $\text{OB} = \nu\text{OD}$,
$\square$ *$\text{AB} = \mu\text{CB}$, and $\square\,\text{CB} = \nu\text{CD}$;
$\therefore\ \square\,\text{AB} = \mu\nu\text{CD}$;
i.e. $\square\,\text{AB} : \square\,\text{CD} = \mu\nu =$ product of ratios of sides.

Cor.—'**The ratio of two squares is the square of the ratio of their sides.**'

* Or $\triangle$ AOB, &c.

The problem of measuring an area is to find how much of unit square is required to make up the area—i.e. to find the ratio of the given area to unit area.

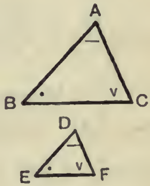
From Theorem 98, if AB is any rectangle and CD unit square, μ, ν are the measures of the sides, and $\mu\nu$ the measure of the rectangle; hence,

'The measure of a rectangle is the product of measures of its sides, and of a square the square of measure of its side.' *

Other areas may be measured by measuring the equivalent rectangle. Thus:

'The measure of a parallelogram is the product, and of a triangle half the product, base × altitude.'

Theorem 99.—'The areas of two similar triangles or two similar polygons are proportional to the squares of corresponding sides.'

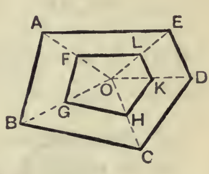


(i.) If tr. ABC ||| DEF,
they have a given ang. $B = E$;

$$\therefore \frac{\triangle ABC}{\triangle DEF} = \frac{AB}{DE} \times \frac{BC}{EF} = \left(\frac{AB}{DE}\right)^2;$$

i.e. $\triangle ABC : \triangle DEF = AB^2 : DE^2$.

(ii.) If the similar polns. AB..., FG... are placed to be similarly situated about O, so that FG ∥ AB, &c.;



$$\text{then } \frac{\triangle OAB}{\triangle OFG} = \frac{OB^2}{OG^2} = \frac{\triangle OBC}{\triangle OGH}$$

$$= (\text{similarly}) \frac{\triangle OCD}{\triangle OHK} = \&c.$$

$$= \frac{\text{sum of } \triangle\text{s } OAB + OBC + \&c.}{\text{sum of } \triangle\text{s } OFG + OGH + \&c.} \text{ (summation)};$$

$$\therefore \text{ area } ABCDE : \text{area } FGHKL = \triangle OAB : \triangle OFG$$

$$= AB^2 : FG^2.$$

Cor.—'Corresponding sides of similar polygons are proportional to the square roots of their areas.'

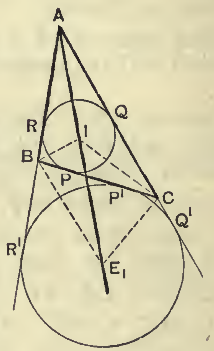
This is important for constructing similar figures of given areas. Thus if areas of similar pentagons are as $3 : 1$, the sides are as $\sqrt{3} : 1$.

* The proof of this suggested in Ch. I. applies to rational numbers only.

Theorem 100.—'If r, r_a are radii of incircle and ecircle in angle A of a triangle, centres I, E_1, then

$$r=\frac{\Delta}{s},\quad r_a=\frac{\Delta}{s-a},\quad rr_a=(s-b)(s-c),$$
$$\Delta=\sqrt{s(s-a)(s-b)(s-c)},\quad \mathrm{AI}\,.\,\mathrm{AE}_1=bc.\text{'}$$

If P, Q, R and P′, Q′, R′ are points of contact of sides; then—



(i.) $\Delta=\mathrm{AIB}+\mathrm{BIC}+\mathrm{CIA}$
$$=\tfrac{1}{2}(rc+ra+rb)=rs;$$
$$\therefore\ r=\frac{\Delta}{s}.$$

(ii.) $\mathrm{E_1R'}\parallel\mathrm{IR}$;
$$\therefore\ \frac{r_a}{r}=\frac{\mathrm{AR'}}{\mathrm{AR}}=\frac{s}{s-a};$$
$$\therefore\ r_a=\frac{rs}{s-a}=\frac{\Delta}{s-a}.$$

(iii.) The bisector $\mathrm{E_1B}\perp\mathrm{IB}$;
$\therefore$ ang. $\mathrm{E_1BP'}$ = compt. of $\mathrm{IBP}=\mathrm{BIP}$;
$\therefore$ triangle $\mathrm{E_1BP'}\ |||\ \mathrm{BIP}$, and $\dfrac{\mathrm{E_1P'}}{\mathrm{BP}}=\dfrac{\mathrm{BP'}}{\mathrm{IP}}$;
$\therefore\ rr_a=\mathrm{BP}\,.\,\mathrm{BP'}=(s-b)(s-c)$.

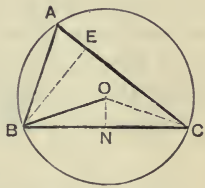
(iv.) $\Delta^2=rs\,.\,r_a(s-a)=s(s-a)(s-b)(s-c)$ by (i.), (ii.), (iii.);
i.e. $\Delta=\sqrt{s(s-a)(s-b)(s-c)}$.

(v.) Ang. $\mathrm{IBE_1}$ = rt. ang. = $\mathrm{ICE_1}$,
$\therefore$ quadl. $\mathrm{IBE_1C}$ is cyclic;
$\therefore$ ang. $\mathrm{IE_1C}=\mathrm{IBC}$ (same arc) $=\mathrm{IBA}$,
and ang $\mathrm{E_1AC}=\mathrm{BAI}$, $\because$ $\mathrm{E_1A}$ bisects ang. A;
$\therefore$ tr. $\mathrm{E_1AC}\ |||\ \mathrm{BAI}$, and $\dfrac{\mathrm{AE_1}}{\mathrm{AB}}=\dfrac{\mathrm{AC}}{\mathrm{AI}}$;
$\therefore\ \mathrm{AI}\,.\,\mathrm{AE_1}=\mathrm{AB}\,.\,\mathrm{AC}=bc$.

Ex. 1. Write down corresponding forms for r_b, r_c.

Ex. 2. Calculate the radii r, r_a, r_b, r_c, and area of a triangle of sides 4 cm., 5 cm., 7 cm. Verify by drawing and measuring.

Theorem 101.—'If R is the circumradius of a triangle, $R=\dfrac{abc}{4\Delta}$.'



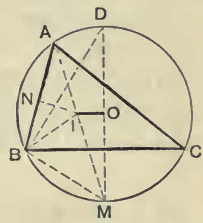
If NO is the rt. bisector of BC,
O the circumcentre,
make BE perp. to AC;
then ang. $BON=\frac{1}{2}BOC=A$ (at circumf.);
$\therefore$ rt. tr. BON ||| BAE;

$$\therefore \frac{OB}{AB}=\frac{BN}{BE}=\frac{BC.AC}{2BE.AC}=\frac{ab}{4\Delta};$$

$$\therefore R=\frac{abc}{4\Delta}.$$

Ex. Calculate R for the triangle of sides 4 cm., 5 cm., 7 cm., and verify by drawing.

Theorem 102.—'If O, I are circumcentre and incentre of a triangle, $OI^2=R^2-2Rr$.'



If the incircle touches AB at N, and AI bisects arc BC of circumcircle at M, and MO meets this circle in D;
then ang. $MIB=IBA+IAB$, int. opp. angs.,
$=IBC+IAC$
$=IBC+MBC$, same arc,
$=MBI$;
$\therefore MB=MI$.

Also, ang. $IAN=BDM$, same arc;
$\therefore$ rt. tr. IAN ||| MDB;

$$\therefore \frac{IA}{MD}=\frac{IN}{MB}=\frac{r}{MI};$$

$\therefore IA.MI=r.MD=2Rr$.

But $IA.MI=R^2-OI^2$, in circumcircle (Th. 93);
$\therefore R^2-OI^2=2Rr$;
i.e. $OI^2=R^2-2Rr$.

Ex. 1. Prove similarly, if E_1 is ecentre in angle A,

$$OE_1{}^2=R^2+2Rr_a.$$

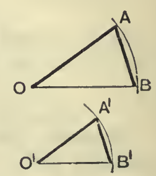
Ex. 2. Measure OI for the triangle of sides 4 cm., 5 cm., 7 cm., and compare with the value calculated from Exx. of Thh. 100, 101.

Theorem 103.—(i.) 'A circle is the limit of a regular in- or circum-polygon when the number of sides is infinite.'

By definition of tangent (Ch. IV., Def. 28), if one angular point of a regular polygon moves up to coincidence with its neighbour, the number n of its sides increasing infinitely, the inpolygon coincides with the circumpolygon, and therefore also with the circle which always comes between them.

We derive two important consequences:

(ii.) 'The ratio $\dfrac{\text{circumference}}{\text{diameter}}$ of a circle (π) is constant.'



If AB, A′B′ are sides of reg. inpolns. of n sides in circles, centres O, O′, diameters D, D′, then trs. OAB, O′A′B′ are similar;

$$\therefore \frac{\text{perimeter of poln. AB}}{\text{D}} = \frac{n\,.\,\text{AB}}{2\,.\,\text{OB}} = \frac{n\,.\,\text{A}'\text{B}'}{2\,.\,\text{O}'\text{B}'}$$

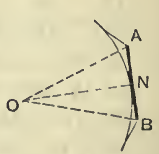
$$= \frac{\text{perimeter of poln. A}'\text{B}'}{\text{D}'}.$$

And each circle is the limit of its inpolygon when the number of sides is infinite;

$$\therefore \frac{\text{circumf. of AB}}{\text{D}} = \frac{\text{circumf. of A}'\text{B}'}{\text{D}'} = \text{constant.}$$

VALUE OF π.—The perimeters of in- and circum-poln. of 16384 sides of diameter 1″ are 3·1415925″... and 3·1415927″..., and π (nearly half-way between these numbers) = 3·1415926...

(iii.) 'The area of a circle is half the product of circumference and radius.' (Area $= \pi r^2$.)



The area of the reg. circumpoln. of n sides AB is

$$n \triangle \text{OAB} = \tfrac{1}{2}\, n\,.\,\text{AB}\,.\,\text{ON}$$

$$= \tfrac{1}{2} \text{ perimeter} \times \text{rad. of circle};$$

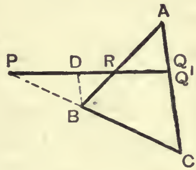
$$\therefore \text{area of circle} = \text{limit of circumpoln.}$$

$$= \tfrac{1}{2} \text{ circumference} \times \text{radius.}$$

Ex. Calculate the circumference and area of a circle of radius 2·9 cm. Also the arcs and areas of sectors of 32°, 72°, 225°, 304° of the same circle.

Some examples are given at the end of Ch. I.

Theorem 104.—'A transversal cutting the sides a, b, c of a triangle in P, Q, R makes $\frac{AR}{BR}\cdot\frac{BP}{CP}\cdot\frac{CQ}{AQ}=+1$.'* (Menelaus' theorem.)



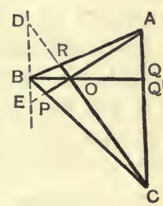
Draw BD parl. to AC.
Then by similar triangles,

$$\frac{AR}{BR}=\frac{AQ}{BD},\quad \frac{BP}{CP}=\frac{BD}{CQ};$$

$$\therefore \frac{AR}{BR}\cdot\frac{BP}{CP}\cdot\frac{CQ}{AQ}=\frac{AQ}{BD}\cdot\frac{BD}{CQ}\cdot\frac{CQ}{AQ}$$
$$=+1.$$

Conversely, '**If P, Q, R satisfy this condition, and PR meets AC in Q′, we can show CQ′ : AQ′ = CQ : AQ ;**
∴ Q coincides with Q′, and hence P, Q, R are collinear.'

Theorem 105.—'**Three concurrent lines OA, OB, OC from the vertices of a triangle cutting the opposite sides in P, Q, R make** $\frac{AR}{BR}\cdot\frac{BP}{CP}\cdot\frac{CQ}{AQ}=-1$.'* (Ceva's theorem.)



Draw DBE parl. to AC. Then the parls. DBE, CQA are similarly divided;

$\therefore \frac{CQ}{AQ}=\frac{DB}{EB}$; and by similar triangles,

$$\frac{AR}{BR}=\frac{AC}{BD},\quad \frac{BP}{CP}=\frac{BE}{CA};$$

$$\therefore \frac{AR}{BR}\cdot\frac{BP}{CP}\cdot\frac{CQ}{AQ}=\frac{AC}{BD}\cdot\frac{BE}{CA}\cdot\frac{DB}{EB}$$
$$=-1.$$

Conversely, '**If P, Q, R satisfy this condition, and AP, BQ′, CR are concurrent, we can show CQ′ : AQ′ = CQ : AQ ;**
∴ Q coincides with Q′, and AP, BQ, CR are concurrent.'

Note. These are simple tests to apply for the collinearity of three points on the sides, or the concurrency of three rays through the vertices, of a triangle.

Ex. Practise these on the 'Solution of Problems' section of Ch. III.

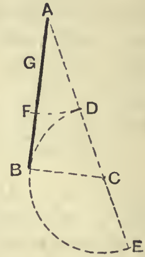
* The ratio AR : BR is positive if the directions from A and B to R are the same, and negative if these directions are opposite.

Square of Mean Section—Radical Axis—Polygon of Given Area and Form.

Construction 37.—'Divide a straight line so that the rectangle of the whole and one part is equal to the square on the other part.' *

If AB is the line, make BC half AB and perp. to it,
draw circle DBE, centre C, to cut AC in D, E; make AF eql. to AD;
then AB touches circ. DBE, and $DE = AB$;

$\therefore\ AF^2 = AD^2 = AD(AE - DE)$
$= AD \,.\, AE - AD \,.\, DE = AB^2 - AF \,.\, AB$
$= AB \,.\, FB.$



Hence AB is divided as required at F.

Construction 38.—'Construct the radical axis of two circles.'

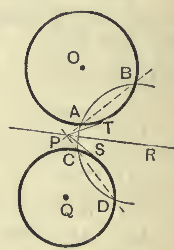
If the circles cut or touch, the common chord or tangent is the radical axis.

If the circles do not meet:

Draw a third circle to cut the given circles, centres O, Q, in A, B and C, D.

Draw AB, CD to meet in P, and PR, the radical axis, perp. to OQ.

If PT, PS are tangents to the circles,
$PT^2 = PA \,.\, PB = PC \,.\, PD = PS^2$;
$\therefore$ P is on radical axis, which is therefore PR.

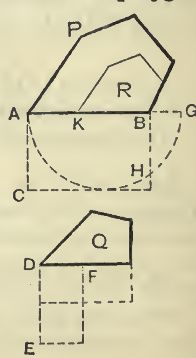


Construction 39.—'Construct a polygon similar to a polygon P and equal in area to another Q.'

On side AB of P make rect. BC eql. to P,
on side DE eql. to AC make rect. EF eql. to Q,
make BG in AB produced eql. to DF,
draw semcle. AHG, make BK eql. to BH.

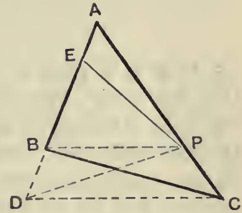
Construct R on BK similar to P;

$\therefore\ R : P = BK^2 : BA^2 = BH^2 : BA^2 = BA \,.\, BG : BA^2$
$= BG : BA = DF : BA = Q : P$;
$\therefore\ R = Q$, and R ||| P.



* Alternatively, construct the mean part BG of AB;
$\therefore\ BG^2 = AG \,.\, AB.$

Construction 40.—'Cut off the nth part of a triangle by a straight line through a given point in one side.'



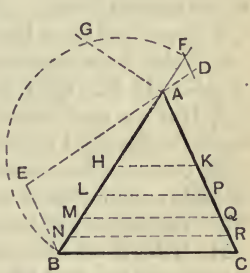
If P is the point in side AC of tr. ABC, make CD parl. to PB;
$\therefore$ $\triangle$ PBD = PBC,
and $\triangle$ PAD = ABC.

Make AE the nth part of AD;

$$\therefore \triangle \text{PAE} = \frac{\triangle \text{PAD}}{n} = \frac{\triangle \text{ABC}}{n}.$$

Note. This construction can be extended to divide a triangle or polygon into n equal parts.

Construction 41.—(i.) 'Cut off the nth part of a triangle by a parallel to one side; (ii.) divide a triangle into n equal parts by parallels to a side.'



If HK parl. to BC makes AHK the nth part of ABC, then AB $= \sqrt{n}$AH,
$\because$ triangle ABC ||| AHK.

(i.) On any line DAE, make AE $= n$. AD on opp. sides of A;
make DF parl. to BE; $\therefore$ AB $= n$AF.

On diameter BF make semicircle FGB, make AG perp. to BF,
$\therefore$ AG = mean propl. of AB, AF;
$\therefore$ AB : AG = AG : AF $= \sqrt{n}$.

Make AH eql. to AG; $\therefore$ AB $= \sqrt{n}$AG $= \sqrt{n}$AH.

Make HK parl. to BC; $\therefore$ triangle ABC ||| AHK;
$\therefore$ $\triangle$ ABC : $\triangle$ AHK = AB^2 : AH$^2 = n$.

Hence AHK is the nth part of ABC.

(ii.) Make in succession along AB, AL = GH, AM = GL, AN = GM,* and so on; and draw LP, MQ, NR, &c. parl. to BC.

Then AL^2 = GH^2 = 2AH^2, AM^2 = GL^2 = 3AH^2, &c.;
$\therefore$ $\triangle$ ALP = 2AHK, $\triangle$ AMQ = 3AHK,
and so on.

Thus HK, LP, MQ, &c. divide the triangle into n equal parts.

* If the work is correct, GN = AB.

EXAMPLES—XXIX.

THEOREMS.

1. If P is a point in a straight line AB, Q in AB produced, then $AP^2+PB^2+2AP.PB=AQ^2+BQ^2-2AQ.BQ$.

2. If M is the mid point of AB in Ex. 1, and AP, AQ > PB, BQ, then $AP.PB=MB^2-MP^2$; $AQ.BQ=MQ^2-MB^2$.

3. Show also that $PA^2-PB^2=4MP.MB$. And write down the corresponding form for the point Q.

4. The sum of squares on the diagonals of a parallelogram is equal to the sum of squares on the four sides.

5. The sum of squares on the diagonals of a quadrilateral is less than the sum of squares on the sides.

6. The sum of squares on the diagonals of a trapezium is equal to the sum of squares on the non-parallel sides and twice the rectangle of the parallel sides.

7. Show that Ex. 4 is a particular case of Ex. 6.

8. If AB is a diameter and CD a chord of a circle, the sum of squares on AC, AD, BC, BD is constant and equal to $2AB^2$.

9. If P is a fixed point inside a circle and AB a chord parallel to the diameter through P, then PA^2+PB^2 is constant.

10. A square has for its side the sum of perpendicular sides of a right triangle; show that it exceeds the square on the hypotenuse by four times the area of the triangle.

11. If M, D are mid point and foot of perpendicular on the side BC of a triangle ABC, and AB > AC, then $AB^2-AC^2=2BC.MD$.

12. The locus of a point whose difference of squares of its distances from two fixed points is constant is a straight line.

13. Circles, centres B, C, are drawn through the orthocentre of a triangle ABC. Show that the tangents from A to the two circles are equal. Show also that the circles meet on the circumcircle.

14. The mid points of the four common tangents of two circles are collinear.

15. A circle whose centre is on the axis of a coaxial system of circles cuts one of them at right angles; show that it cuts all of them at right angles.

16. A straight line cuts the axis of a coaxial system of circles in V, and the circles in PP′, QQ′, &c.; show that VP . VP′ = VQ . VQ′, &c.

17. If a point P is joined to the vertices of a parallelogram ABCD, and AC > BD, then $PA^2+PC^2-(PB^2+PD^2)=\frac{1}{2}(AC^2-BD^2)$.

18. A jointed rhombus ABCD moves so that one diagonal AC is always a chord of a fixed circle, centre O. Show that BD always passes through O, and that OB . OD is constant.

19. The greatest rectangle of the two parts into which a given straight line can be divided internally is the square on half the line.

20. A square has the greatest area of all rectangles of a given perimeter.

21. A square has the least perimeter of all rectangles of a given area.

22. Are Ex. 20 and Ex. 21 true if parallelogram is written for rectangle?

23. If ABCD are four points in order in a line, and AC is the mean proportional of AB, AD, then $AB : AD = AB^2 : AC^2$.

24. If four straight lines are proportional, similar polygons on the first two are proportional to similar polygons on the last two.

25. Show that Pythagoras' theorem is true if similar polygons are used instead of squares. (See Ch. III.)

26. A diameter AB of a circle is divided into two parts at C, and semicircles described on AC, CB as diameters, on opposite sides of AB; show that the sum of arcs AC + CB is half the length of the first circle.

27. Show also that the two areas into which the arcs AC, CB divide the whole circle are proportional to the diameters AC, CB.

28. Deduce from Ex. 27 a construction for dividing a circle by semicircular arcs into n equal parts.

29. Similar sectors of circles are proportional to the squares of their radii or chords.

30. Similar segments of circles are proportional to the squares of their radii or chords. (Treat as the sums or differences of similar sectors and similar triangles.)

31. Show that Pythagoras' theorem is true if similar segments of circles are used instead of squares.

32. Show that the centres of similitude of three circles lie three by three on straight lines. What figure is formed by the straight lines?

33. Tangents to the circumcircle of a triangle at its vertices meet the opposite sides in collinear points.

34. If P, Q, R are collinear points on the sides BC, CA, AB of a triangle, and P, P′ divide BC harmonically, show that AP′, BQ, CR are concurrent.

35. If lines AP, BQ, CR to the sides of a triangle ABC are concurrent, and P, P′ divide BC harmonically, show that P′, Q, R are collinear.

36. If a circle cuts the sides BC, CA, AB of a triangle in pairs of points P, P′; Q, Q′; R, R′; then if AP, BQ, CR are concurrent, so also are AP′, BQ′, CR′.

EXAMPLES—XXX.

CONSTRUCTIONS.

1. Divide a straight line 3·24″ so that the rectangle of the whole line and one part is equal to the square on the other. Measure.

2. Give a construction for lines of $\sqrt{2}''$, $\sqrt{3}''$, $\sqrt{4}''$, &c. from a line of 1″.

3. Construct a line whose ratio to a given line is $\sqrt{5}+1$.

4. Calculate the altitude of an equilateral triangle in terms of a side. Calculate the area of an equilateral triangle of 1·73″ side. Calculate the area of an isosceles triangle, $a=b=5$ cm., $c=6$ cm.

5. Construct a triangle, $a=1{\cdot}8''$, $b=2{\cdot}4''$, $c=2{\cdot}7''$. Calculate and measure its area.

6. Calculate the area and radii of the circles of a triangle, $a=12$ ft., $b=16$ ft., $c=18$ ft. (Tabulate s, $s-a$, $s-b$, $s-c$, Thh. 100, 1.)

7. Construct the radical axis of two circles, radii $\frac{3}{4}''$, $1\frac{1}{2}''$, distance of centres 3″.

8. Construct a point whose tangents to two circles, radii 3·2 cm., 2·5 cm., join of centres 4 cm., are 2 cm. long. How many points?

9. Construct the side of a square equal to the difference of squares on 3·5 cm. and 2·8 cm.

10. Construct the locus of a point whose difference of squares of its distances from two fixed points 5 cm. apart is 4 sq. cm.

11. Find a point in a straight line 3″ long whose difference of squares of its distances from the ends is 1 sq. in.

12. Construct the locus of a point P whose sum of squares of its distances from fixed points A, B is constant. (See Th. 91.)

13. Find a point in a line 6 cm. long such that the sum of squares of its distances from the ends is 26 sq. cm.

14. Construct the locus of all points the sum of whose squares from the ends of the line in Ex. 13 is 26 sq. cm.

15. How far can you see out to sea from a height (of the eyes) of (i.) 10 ft., (ii.) 1000 ft., earth's radius 3960 miles?

16. Construct the side of a square of 3·76 sq. in. Measure.

17. Construct a square equal to (i.) 3 times, (ii.) $\frac{1}{3}$ of a given square.

18. Construct a regular pentagon of 1″ side; and a similar pentagon of double the area. (Calc. the new side, Th. 99, Cor.)

19. Construct a regular hexagon having an area of 1 sq. in. (Make any hexagon, and adapt Constr. 39.)

20. Draw an isosceles triangle, $a=b=1{\cdot}73''$, $C=51°$; take a point on CA, 1″ from C, and draw a line through it bisecting the triangle.

21. Bisect a quadrilateral by a line through a vertex. Write down the construction for cutting off a sixth part.

22. Construct an equilateral triangle whose area is 12 sq. cm. Verify by measuring. (Adapt Constr. 39.)

23. Divide a straight line into two parts whose squares have a given ratio, say 3 : 5. (See Constr. 13, Note, for constr. of $\sqrt{3}$, $\sqrt{5}$.)

24. Divide a straight line so that the sum of squares on the whole and one part is three times the square on the other part. (Mean section.)

25. Draw an equilateral triangle, side 2·7 cm., and divide into four equal parts by parallels to one side. Show that Construction 41 can be shortened in this case.

26. Draw a triangle, $a=1{\cdot}72''$, $b=1{\cdot}31''$, $c=1{\cdot}56''$. Take AP on AB $=1''$, draw a line through P to cut off one-fifth of the triangle.

27. Construct a triangle, $a=3{\cdot}5$ cm., $b=4{\cdot}2$ cm., $c=2{\cdot}8$ cm., and cut off a fifth part by a parallel to the side b. Which side gives the easiest construction?

28. Calculate the area of a regular hexagon on a side of 1″. What is that of a regular hexagon of 2″ side?

29. Find the ratio of the areas of the circumscribed square and inscribed hexagon of a circle.

30. Divide the arc of a semicircle into two parts, the squares of whose chords have the ratio 2 : 3.

31. Divide a straight line 3·8 cm. long into two parts whose rectangle is the greatest possible. What is the area of this rectangle?

32. Make a square equal to five times the square of Ex. 31.

33. Show how to derive in succession the complete series of squares of areas 1, 2, 3, 4... What series of numbers do their sides represent?

34. Construct the radical axis of two non-intersecting circles, centres O, Q, to meet the line of centres in N.

Construct a third circle coaxial with the first two, and passing through a fixed point A on OQ.

35. In the construction of Ex. 34, determine the two points D, D_1 on OQ such that $ND=ND_1=$length of tangent from N to the circles.

36. Construct the circles through the points D, D_1 of Ex. 35, coaxial with the original circles. What do you find? (These circles are the limiting circles or foci of the system of coaxial circles.)

37. Can you construct the limiting circles of a coaxial system when the circles cut in two points A, B? Why?

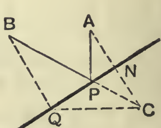
38. Two regular pentagons have sides 3 cm., 5 cm. Construct a square whose area is the sum of their areas.

The greatest and least values of a varying quantity can sometimes be found readily by a simple application of a known inequality; the most general method is that of Thh. 85, 86, Ch. IV., in which some variation of a given form or value increases or diminishes the quantity in question. We give a few examples; others will be found in Ex. XXXIV., 114–133.

Construction 42.—'Find the shortest path between two points A, B to touch at a given line PQ.'

If P is the point where the shortest path BPA touches PQ, then BP and AP are straight, as otherwise the path may be shortened.

Make ANC perp. to PQ, NC eql. to AN, join BC to cut PQ in P, join AP.



Then NP is rt. bisector of AC;

∴ PC = PA, and path BPA = BC.

If BQA is any other path,

BQ + QC, i.e. BQ + QA > BC, in triangle BQC;

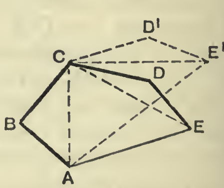
i.e. path BQA > BPA;

∴ BPA is the shortest path.

Note. If A, B are on opposite sides of the line PQ, then AB is the shortest path.

Theorem 106.—'If all the sides of a polygon ABCDE are given except one AE, the greatest polygon is that in which the last side AE subtends right angles at the other vertices B, C, D.'

If ACE is not a right angle, rotate triangle CDE about C to make CE perp. to AC;



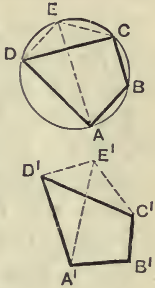
∴ alt. of E from CA increases, base CA remains unaltered, and area of tr. ACE and of the poln. increases;

∴ if angle ACE is not a rt. ang., area of poln. ABCDE can be increased;

∴ the maximum polygon has ang. ACE, and similarly ABE and ADE, each a right angle.

Ex. The maximum quadrilateral with three equal sides a given, is half the regular hexagon on side a.

Theorem 107.—'Of all polygons of given sides, the greatest is that which can be inscribed in a circle.'



If ABCD is inscribed, and A′B′C′D′ having the same sides cannot be inscribed, in a circle, draw diameter AE;

make triangle D′E′C′ congruent to DEC.

Then the circle on diam. A′E′ does not pass through all the points B′, C′, D′; not through C′, say.

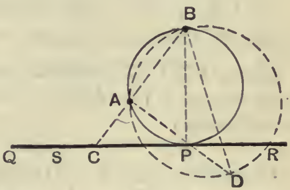
Also, quadls. ABCE, A′B′C′E′ having three sides the same, ABCE in a semicircle $>$ A′B′C′E′.

Similarly ADE $\geqq$ A′D′E′;

∴ poln. ABCED $>$ A′B′C′E′D′, and ABCD $>$ A′B′C′D′.

Construction 43.—'If A, B are given points, PQ a given line, find the points in PQ at which AB subtends (i.) maximum, (ii.) minimum angles.'

If AB meets PQ in C, and A, B are on the same side of PQ, and if circles through A, B touch PQ in P and Q respectively;



take R any other point in CP, and draw circle ARB cutting AP in D;

∴ ang. APB $>$ ADB, int. opp. ang.,

$>$ ARB, same arc;

∴ APB is a maximum on one side of C.

Similarly, AQB is a maximum on the other side;

and ACB (= zero) and the angle of the parallels to PQ from A, B (= zero) are minima.

This is a good example of maxima and minima occurring alternately.

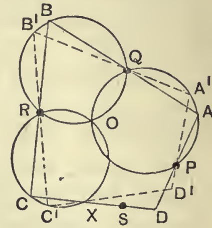
Discuss the case when A, B are on opp. sides of PQ.

Ex. 1. Find the point P within a triangle ABC such that PA + PB + PC is a minimum.

Ex. 2. An isosceles triangle has a less perimeter than any other triangle of the same base and area.

Ex. 3. An equilateral triangle has a less perimeter than any triangle of equal area.

Construction 44.—'**Circumscribe a quadrilateral of given form * to a given quadrilateral PQRS.**'



If $\alpha\beta\gamma\delta$ has the given form, on PQ, QR describe circles whose arcs PAQ, QBR have angles α, β, and which meet in O.

Draw chd. A′QB′, make B′RC′ such that B′C′ : B′A′ = $\beta\gamma : \beta\alpha$;

∴ triangle A′B′C′ ||| $\alpha\beta\gamma$.

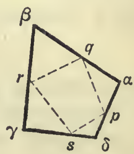
Make circ. ORC′, and ang. ROX = suppt. of γ; draw C′X, A′P to D′.

∴ C′ = γ, hence D′ = δ.

Draw in order SXC, CRB, BQA, APD.

Then tr. OAB ||| OA′B′, OBC ||| OB′C′;

∴ tr. ABC ||| A′B′C′ ||| $\alpha\beta\gamma$. (Th. 39, Ch. III.)



And ang. DCA = γ − BCA = $\delta\gamma\alpha$, DAC = $\delta\alpha\gamma$;

∴ tr. DCA ||| $\delta\gamma\alpha$;

∴ quadl. ABCD ||| $\alpha\beta\gamma\delta$, and circumscribes PQRS.

A very simple construction serves for a parm., rect., or sq.

If QX′ makes ang. of parm. with PR, and QX′ : RP = ratio of sides, SX′ fixes one side.

Note. A′B′C′D′ ||| ABCD, and can be rotated about O to be similarly situated to ABCD, and then multiplied (by OA : OA′) into coincidence.

Construction 45.—'**Inscribe a quadrilateral of given form in a given quadrilateral ABCD.**'

If *pqrs* is the given form, circumscribe $\alpha\beta\gamma\delta$ similar to ABCD; divide the sides of ABCD in the same propn. at P, Q, R, S.

Theorem 108.—'**If three sides of a polygon of given form pass through fixed points P, Q, R, any fourth side traverses a fixed point X.**'

The fourth side CD completes a quadl. ABCD of given form.

Draw circs. PAQ, QBR meeting in O, ORC meeting CD in X;

∴ X is a fixed point; and if A′B′C′D′ ||| ABCD,

C′ is on circ. ORC, and ang. BC′X = C = C′;

∴ C′D′ coincides with C′X, and traverses fixed point X.

* That is, similar to a given quadl.

Construction 46.—'Construct a triangle ABC of given form* so that, A being given, B, C are on given curves.'

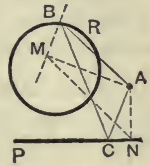
If $\alpha\beta\gamma$ has the given form, and A is a vertex given, e.g. on some third curve:

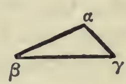
(i.) If one curve is a straight line P, make AN perp. to P, tr. AMN simr. to $\alpha\beta\gamma$, MB perp. to AM, to meet the curve R at B; make ang. BAC eql. to α, to meet P at C.

Then rt. tr. BAM ||| CAN,

$\because$ ang. BAM $= \alpha -$ MAC $=$ CAN;

$\therefore$ tr. BAC ||| MAN ||| $\alpha\beta\gamma$. (Th. 39.)

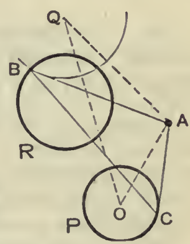




(ii.) If P is a circle, centre O, rad. r; make tr. AQO simr. to $\alpha\beta\gamma$, draw arc B, cent. Q, rad. $r \times \dfrac{\alpha\beta}{\alpha\gamma}$, to cut curve R in B; make ang. BAC $= \alpha$.

Then tr. ABQ ||| ACO† (Th. 41, i.);

$\therefore$ tr. ABC ||| AQO ||| $\alpha\beta\gamma$. (Th. 39.)



In each case the curve P is rotated through α about A and then multiplied by the ratio $\alpha\beta : \alpha\gamma$ into a similar curve cutting R in the point B.

The method applies to any curve P.

EXAMPLES—XXXI.

1. Inscribe a triangle of given form in a semicircle. (Choose A some point on the circumference, and use the diameter as the curve P and the circle as R.)

2. Describe a triangle, A$=60°$, $b : c = 2 : 1$, with its vertices on (i.) three parallel lines, (ii.) three concurrent lines, (iii.) three sides of a triangle. (Choose A on one line.)

3. Describe an isosceles right triangle with the right-angled vertex fixed and the others on given lines.

4. Describe an equilateral triangle (i.) with two vertices on a given circle, and one on a given tangent; (ii.) with its vertices on three concentric circles.

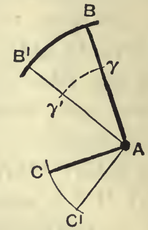
5. Circumscribe to a square a parm., ang. 60°, sides 8 : 7.

* That is, similar to a given triangle.

† If the right point C is chosen on the line AC.

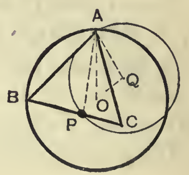
The principle underlying Construction 46 is useful for constructing certain loci.

Theorem 109.—'If a triangle of given form is rotated about a fixed vertex A, and a second vertex B describes any curve, the third vertex C describes a similar curve.'



If ABC, AB′C′ are two posns. of the triangle, BB′, CC′ are corresponding chds. of their curves, and triangle ABB′ ||| ACC′;
∴ chd. CC′ and curve of C can be rotated through ang. A into posn. $\gamma\gamma'$, similar and similarly situated to BB′ and the curve of B.

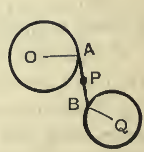
Ex. 'On chords AB through a fixed point A of a circle, equilateral triangles ABC are drawn; find the locus of a point dividing BC in a given ratio.' (Take 2 : 1.)



In triangle ABP, BP : AB = BP : BC = const.,
and ang. B = 60°;
∴ triangle ABP has a given form;
and it rotates about A, B describing a circle;
∴ P describes a similar circle.
Make triangle AOQ simr. to ABP;
∴ tr. APQ ||| ABO;
∴ QP : OB = QA : OA = const.;
∴ Q is centre of circle.

Plotting Loci.

Loci not straight lines or circles can be drawn by plotting any number of points, and drawing a freehand curve through them.



Ex. The ends of a rod AB move round two circles, centres O, Q; plot the locus of a point P on AB.

Take OA 1″, QB $\frac{3}{4}$″, OQ $3\frac{1}{2}$″, AB 3″, AP 1″.

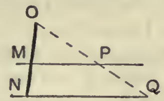
Set the divider to AB (3″), mark two points A, B on the circles, bring a straight-edge to them; prick P, 1″ from A.

Take a fresh position and repeat, going carefully round one circle, until the form of the locus is clear.

The curve is like a distorted 8. But for different values of the radii and length of rod, and for different positions of P, the curve may have other forms.

In plotting curves where several pairs of lines in a fixed ratio μ are required, this construction is useful:

Make OM : ON = the given ratio μ; parls. MP, NQ give OP : OQ = μ.



Definition 36.—A **conic** is the locus of a point whose distance from a fixed point has a constant ratio to its distance from a fixed line.

The fixed point is the **focus**, the line the **directrix**, and the ratio the **eccentricity** of the conic.

A conic is a parabola, ellipse, or hyperbola, according as eccentricity $= 1$, < 1, > 1.

EXAMPLES—XXXII.

1. Plot a parabola; an ellipse, ecc. 2:3; a hyperbola, ecc. 3:2; focus—directrix, 1″. (Use squared paper.)

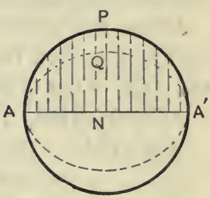
2. Plot* the locus of P when the sum of distances PA + PB from fixed points A, B is constant. (An ellipse, foci A and B. It is a closed oval curve.)

3. Plot the locus of P when PA − PB is constant. (A hyperbola, foci A, B.)

4. Draw a circle, draw perpendiculars PN to a diameter AA′; multiply these by a constant ratio QN : PN $= \mu$ (say 2 : 3). Plot the locus of Q.

5. In Ex. 4, show that $QN^2 : AN \,.\, NA' = \mu^2$.

This is an ellipse whose axes are AA′ and μAA′. It represents the *plan* of a circle tilted up from the horizontal.



ENVELOPES.

If a straight line moves according to some law—e.g. the side BC of the triangle in Th. 109—a curve, the **envelope** of the line, exists to which the straight line is a tangent. If a number of positions are drawn sufficiently near, the form of the envelope is shown; and the curve can sometimes be found by our knowledge of geometry.

On the next two pages are given examples of methods of finding the point of contact of an envelope with its generating line in any position.

* This should also be drawn with divider, a loop of cotton, and pencil.

Ex. Plot the envelope of one side of a right angle whose vertex moves on a circle, and whose other side traverses a fixed point.

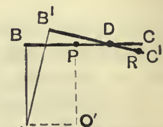
The form of the curve is readily recognised from the figure as an oval. It is an ellipse with the fixed point A as one focus, and H at the same distance from the centre on AO for the other.

If A is external, the envelope is a hyperbola.

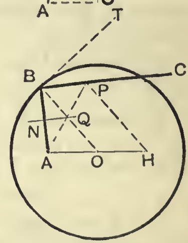
Note. If BC, B′C′ are two near positions of the moving line, meeting in D, and touching the envelope at P, R;
then, as the tangt. B′C′ moves to coince. with BC, R and D move to coince. with P.

Thus the point of contact P on BC has the limiting posn. of the intersection of two coincident tangents.

The envelope is found as the locus of this point.



Thus, in the above example,
angs. B, B′ in two near positions are equal;
∴ A, B, B′, D (2nd fig.), are concyclic; and as B′C′ moves to coince. with BC, BB′ becomes tangt. to the first circle, and to circ. ABP; these therefore touch.



Hence the centre of circle ABP is Q on OB (3rd fig.), and on the rt. bisector of AB;
and AQP is the diameter of circ. ABP.

Also, OQ ∥ HP, ∵ OQ bisects AP, AH;

$$\therefore\ PA + PH = 2QA + 2OQ = 2QB + 2OQ = 2OB$$
$$= \text{constant};$$

∴ locus of point of contact P is an ellipse, foci A, H.

If A is outside the circle, $PA \sim PH$ is constant, and the envelope is a hyperbola.

Note. A line BC of given length (2nd fig.) can be moved into position B′C′ by turning about a centre of rotation O′ (intersection of rt. bisectors of BB′, CC′, or of circs. BB′D, CC′D);
then O′D bisects ang. BDC′.

If now B′C′ turns about O′ to coincidence with BC,
D moves to P, point of contact of BC with envelope,
and O′D to O′P, perp. to tangt. BP.

Hence the point of contact of a moving line in any position with its envelope is the foot of perpendicular on the line from the centre of rotation.

This centre of rotation is not in general fixed, but varies with the position of the line; it is therefore called the **instantaneous centre** of rotation for the particular position of the moving line.

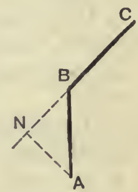
In fig. 3, last page, if E is the point of the side BA of the right angle, which momentarily coincides with A;
then as B begins to move on circle O,
E moves at first along direction AB;
∴ inst. centre O′ is intersection of BO′, AO′, perp. to BT, AB.

And O′P, perp. to BC, determines pt. of contact P.

The method is useful for a figure given in magnitude and form.

By the aid of Theorem 109 we can show that

'The envelope of one side of an angle whose vertex describes a fixed circle, and whose other side traverses a fixed point, is an ellipse or hyperbola.'

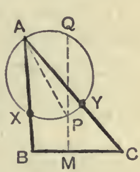


For if ABC is the angle, A the fixed point,
and AN ⊥ BC;
the triangle ANB has a given form, and turns about one vertex A, whilst another vertex B describes a circle;
∴ the third vertex N describes a circle (Th. 109);
∴ envelope of NB—i.e. of BC—is an ellipse or hyperbola.

EXAMPLES—XXXIII.

1. The envelope of a line at a fixed distance from a given point is a circle.

2. If the sides AB, AC of a given triangle traverse fixed points X, Y, the locus of A is a circle, and the envelope of BC another circle. (Bordillier.)



If AP is a diameter of the circle XAY, and PM perp. to BC meets this circle in Q; (P is inst. centre);
Q is a fixed point, and the centre of the envelope.

3. The envelope of one side of a right angle whose vertex describes a straight line, and whose other side traverses a fixed point, is a parabola.

4. Plot the envelope of Ex. 3. Distance focus—line ½″.

5. The vertex of an angle describes a straight line, and one side passes through a fixed point; show that the other side envelopes a parabola.
(Use Ex. 3 and Th. 109, as for ellipse above.)

6. Plot the envelope of the side BC of an equilateral triangle of which A is fixed and B moves on a circle.

What is the envelope when A is on the circle?

EXAMPLES—XXXIV.

General.

1. A straight line is determined in position by one point and the angle it makes with a given direction.

2. Take a point P one inch from a given line, and draw through it a straight line to make an angle of 54° with the line.

3. The right bisector of one side of a rectangle is the right bisector of the opposite side.

4. If opposite sides of a parallelogram have the same right bisector, the angles are right angles.

5. Draw a square of 3·7 cm. side. Without pen or pencil, mark the bisectors of angles and right bisectors of sides.

6. Two straight lines in a plane parallel to a third are parallel to one another.

7. A straight line perpendicular to one side of a parallelogram is also perpendicular to the opposite side.

8. The angle A of a parallelogram ABCD is 63°; a straight line EF cuts AB at an angle of 97°, towards the same part as A; what angle does EF make with AD? with DC? with BC?

9. Draw a parallelogram ABCD, AB = 2 cm., AD = 3 cm., ang. A = 70°. At C outside the parm. make ang. BCE = 110°. Show that DC, CE are in a straight line.

10. A straight line OP through a fixed point O meets a fixed straight line ABC in A. It is then turned in the plane through an angle X, into the position OQ, cutting ABC in B. If C is in AB produced, show that X = PAC − QBC.

11. Draw two lines not meeting on the paper. Measure the angle between them. (Draw a line AB to cut them and apply Ex. 10.)

12. Through a point P on one of the two lines of Ex. 11 draw a parallel to the other, and measure the angle thus formed. What do you notice?

13. Show that the angle of two lines X, Y may be measured by the angle of X, Z, if Z ∥ Y.

14. A straight line makes with two straight lines cutting it angles of 95° and 87° towards the same parts. Calculate their angle.

15. Calculate the exterior angles of a hexagon having five angles 135°, 147°, 161°, 108°, 83°. Sum them.

16. Draw the pentagon ABCDE, given AB = 1″, B = 135°, BC = 1·41″, C = 105°, CD = 3″ = DE, EA = 1·41″. Measure the other angles.

17. Construct a triangle, $a=3{\cdot}72''$, $B=99°$, $C=41°$; draw a line DEF cutting AB, AC, BC in D, E, F, making ang. ADE $=60°$. Calculate angles AED, BFD; and test by measuring.

18. The bisectors of consecutive angles of a quadrilateral meet at E; show that their angle is half the sum of the other angles of the quadl.

19. Construct a triangle, given $a=2{\cdot}7$ cm., exterior angles at B and C, 108° and 152°.

20. Straight lines AD, AE to the side BC of an isosceles triangle make equal internal angles with the equal sides AB, AC. Show that ADE is isosceles.

21. If BE, CD make equal angles BEC, CDB with equal sides AB, AC of a triangle, show that BE$=$CD.

22. Construct a triangle, $a=3{\cdot}72''$, $b=2{\cdot}59''$, $c=2{\cdot}23''$. Draw the right bisector of AC, meeting BC in D. Prove that BD$=a-$AD.

23. ABC is a triangle having $b=4{\cdot}6$ cm., $c=3{\cdot}7$ cm., $A=150°$. The right bisectors of b, c meet a in D, E. Show that $a=$sum of sides of ADE.

24. One only parallel can be drawn to a straight line through a given point.

25. If a right triangle is turned in a plane so that its right-angled sides are interchanged in position, the two positions of the hypotenuse are perpendicular.

26. Deduce from Ex. 25 a construction for a perpendicular by set-square.

27. If two lines a, b are perpendicular respectively to two lines c, d, the angle ab is equal to the angle cd.

28. Given the diagonal $2{\cdot}12''$ of a square, construct the square.

29. The projection of any straight line on a non-parallel line is less than the original line.

30. If A is the greatest angle of a triangle ABC, a line DE, cutting the sides AB, AC in D, E, is less than BC.

31. Construct a rhombus, diagonals $3{\cdot}2''$ and $2{\cdot}8''$. Calculate the length of a side.

32. If M is the mid point of a straight line AB, P any point on it, MP is half the sum or difference of MA and MB, according as P is external or internal.

33. Perpendiculars from two vertices of a triangle on the opposite sides form an angle equal to the third angle of the triangle.

34. Construct a parallelogram, sides $2{\cdot}32''$ and $1{\cdot}76''$, one diagonal $1{\cdot}97''$. Draw a parallel to one side through the diagonal point. Verify that the two parallelograms formed are congruent.

35. Show that a square is divided into eight congruent triangles by bisectors of angles and right bisectors of sides.

36. Show that if straight lines PQ, RS bounded by the two pairs of opposite sides of a square are perpendicular, they are also equal.

37. Construct a quadrilateral with equal diagonals 2·7 cm. long perpendicular to and trisecting each other; draw a rectangle whose sides pass through its vertices.

38. Show that if the diagonals of a quadrilateral are equal and perpendicular, every circumscribing rectangle is a square.

39. Construct a square to circumscribe a given quadrilateral.

40. If PQR is a triangle entirely inside another ABC, show that $p+q+r<a+b+c$.

41. If PQR... is a polygon, with no re-entrant angle, entirely inside another ABC..., the perimeter of PQR... is less than that of ABC...

42. Draw any pentagon, and copy it by ruler and compass.

43. The sum of two sides of a triangle is greater than twice their bisector of angle.

44. If in a trapezium the angles at the ends of one of the parallel sides are equal, the non-parallel sides are equal; and if the non-parallel sides are equal, the angles at the ends of either parallel side are equal; also in either case the diagonals are equal.

45. If the diagonals of a trapezium are equal, the non-parallel sides are equal.

46. If the diagonals or non-parallel sides of a trapezium are equal, the trapezium is cyclic; and the median line of the parallel sides passes through the centre of the circumcircle.

47. Divide a straight line AB at a point P so that PA : PB = the ratio of any two given lines X : Y.

48. Show that there is one only internal point in AB which can divide AB as in Ex. 47; and similarly one only external point.

49. Given $a=4$ cm., $\mathsf{A}=70°$, $b:c=5:3$, construct the triangle.

50. All straight lines through a point P and cutting two parallels in A, B are divided at P in the same ratio PA : PB.

51. Draw two parallels 5 cm. apart, and construct the locus of a point P dividing in the ratio 3 : 4 any line bounded by the parallels.

52. If A′ is the mid point of BC in triangle ABC, and CD is perpendicular to the bisector of angle A, DA′ is parallel to AB, and is equal to half the difference of AB and AC.

53. If AD, AP are perpendicular and bisector of angle A of a triangle, the angle DAP is half the difference of B, C.

54. If one of the parallel sides of a trapezium is fixed, and the altitude and magnitude of the opposite side are given, construct the locus of the diagonal point.

55. The locus of a point the sum of whose distances from two fixed lines is constant is four parallels to the bisectors of angle of the lines.

56. The locus of a point whose distances from two sides of an angle have a fixed ratio X : Y is two straight lines through the angle.

57. Construct a point in a triangle whose distances from the sides are proportional to those sides ($a=5$ cm., $b=6$ cm., $c=8$ cm.).

58. The mid points of sides of a triangle form a similar triangle, similarly situated to the original triangle.

59. Given in position an angle A of a triangle and the mid point of the opposite side, construct the triangle.

60. Construct a straight line through a given point such that two given pairs of parallels cut off equal parts from it.

61. If the equal parts of Ex. 60 are PP′, QQ′ in order, find the locus of the mid point of PQ′ for all positions of the given point.

62. A parallel to a side of a parallelogram cuts the diagonals in P, Q; show that the join of the mid point of PQ to the diagonal point is parallel to another side.

63. If BE is the fourth part of diagonal BD of parallelogram ABCD, and AE meets CD in F, then FC = 2CD.

64. Draw a parallelogram; and inscribe in it a rhombus, given the length of one diagonal.

65. If three vertices of a parallelogram whose sides are parallel to fixed directions move on three fixed straight lines, the fourth vertex moves on a straight line.

66. Inscribe in a given quadrilateral a parallelogram, given the directions of sides. (Use Ex. 65.)

67. Construct a parallelogram, two opposite vertices being fixed, and the other two lying on a fixed circle.

68. Similar segments of circles on equal bases are congruent; or, similar arcs of circles on equal chords are congruent. (Similar segments or arcs have equal angles.)

69. Two triangles ABC, DEF have BC = EF, ang. A = D; show that the vertices of DEF can be placed on the circumcircle of ABC.

70. Given a, B, and the distance of the circumcentre from BC, construct the triangle.

71. Construct a point at which the three sides of a triangle subtend equal angles.

72. If a jointed quadrilateral ABCD has AB fixed, find the loci of C, D and of the mid points of AC, BD.

73. Inscribe a given square in a given square.

74. If H is the orthocentre of ABC, then A, B, C are orthocentres of HBC, HCA, HAB.

75. The circumcentre of a triangle is the orthocentre of the median triangle; and the triangles have a common centroid.

76. If O, H are circumcentre and orthocentre of a triangle, and X the mid point of BC, show that $OX = \frac{1}{2}AH$. And if AH meets BC in D and the circumcircle in K, then $DK = DH$.

77. Construct a triangle, given in position the circumcentre, orthocentre, and an angular point.

78. If G, O, H, N are centroid and circum-, ortho-, N-centres of ABC, and X, Y, Z, L, M, N mid points of sides and of AH, BH, CH; show

(i.) Triangles XYZ, LMN are similarly situated to ABC, about G, H.

(ii.) LX is a diameter of the N-circle and ∥ AO.

(iii.) If AO meets BC in P, the circle, diam. AP, touches the N-circle.

79. The centroid, circum-, ortho-, and N-centres are collinear.

80. Construct a triangle, given the N-circle, circumcentre, and direction of one side.

81. If straight lines AD, BE to the opposite sides of a triangle trisect each other, they are medians.

82. Two straight lines from two vertices of a triangle to the opposite side cannot each bisect the other.

83. Construct a triangle, given angle C and the medians through A, B.

84. The incentre and ecentres of a triangle are each the orthocentre of the triangle of the other three.

85. The orthocentre and angular points of a triangle are the incentre and ecentres of the pedal triangle.

86. Given the ecentres, or two ecentres and incentre, construct the triangle.

87. The circumcircle of a triangle bisects the join of an in- and e-centre.

88. Construct a triangle, given circumcircle, incentre, and a vertex.

89. If AP is the bisector of angle A of a triangle, I the incentre, show that $AI : IP = b + c : a$, and hence construct a point S in AI such that $AS : SI : IP = b : c : a$.

90. If two altitudes of a triangle are given, show that the ratio of the corresponding sides is given.

Construct a triangle, given the altitudes.

91. There is one only point P in a triangle such that angle $PAB = PBC = PCA$, and one only point P′ such that $P'BA = P'CB = P'AC$. (P and P′ are positive and negative **Brocard points.**)

92. If P and P′ are Brocard points of a triangle, show that $PAB = P'BA$, &c. (**Brocard angle.**)

93. Construct the Brocard point P of an isosceles right triangle ABC, right-angled at A, and show that CPA is a right angle.

94. If P, P′ are Brocard points of an isosceles triangle whose angles B, C are each double of A, show that

(i.) Ang. PBA = PCB, ang. APB = 108° = CPB.

(ii.) PC, PB are mean parts of PB, PA, and hence PB + PC = PA.

95. If a is the mean part of b, and b of c, then $a+b=c$.

96. If AC is the mean part of AB, then $AB^2+BC^2=3AC^2$.

97. The circles circumscribing the four triangles of a complete quadrilateral are concurrent.

98. Construct four concurrent circles to meet, two and two, in four given points (e.g. circles A, B meet in P; B, C in Q; C, D in R; D, A in S).

99. If M is the mid point of arc AB of a circle, and a straight line MP cuts the chord in P and the circle in Q, then MA is tangent to circle APQ.

100. If the tangent AC at the end of a diameter AB of a circle is equal to the diagonal of the square on AB, the centroid of ABC is on the circle.

101. If AB is a diameter of a circle, PQ perpendicular to AB from a point P on the circle, and R is taken on AB so that AR = AP; show that

(i.) PR bisects angle QPB;

(ii.) QR is radius of circle touching PQ, QB and arc PB.

102. Inscribe a circle in a part of a semicircle cut off by a perpendicular to the diameter. (Use Ex. 101, ii.)

103. How many revolutions does a 28″ bicycle-wheel make in a ten-mile ride? ($\pi=3{\cdot}1416$.)

104. If ABCD is a cyclic quadrilateral, the rectangle of diagonals is equal to the sum of rectangles of opposite sides. (Ptolemy's theorem. Draw AE to diag. BD, making ang. BAE eql. to CAD. Triangle BAE ||| CAD, DAE ||| CAB.)

105. If ABCD is a non-cyclic quadrilateral, the rectangle of diagonals is less than the sum of rectangles of opposite sides.

106. Construct tangents to a circle from a point without using the centre of the circle.

107. If P is a point on one of two circles, centres O, Q cutting in A, B, PN perpendicular to AB, and PT tangent to the other circle cuts AB in C and the first circle in R, then PC(PT + RT) = 2PN . OQ.

108. Draw a circle to touch a given circle, a tangent to it, and a line through the point of contact.

109. Find the point in a straight line at which two given points not on the line subtend the greatest angle.

110. The sum of squares on the medians of a triangle is $\frac{3}{4}$ of the sum of squares on the sides.

111. The sum of the two lunes formed by semicircles on three sides of a right triangle, on the same side of the hypotenuse, is equal in area to the triangle. (Apply Pythagoras' theorem.)

112. Cut off the nth part of a quadrilateral by a line through a point in one side.

113. Draw a circle of given radius to cut two given circles orthogonally.

114. The greatest distance from the centre of a circle of a point P, from which lines PQ, PR drawn to the circle are perpendicular, is $\sqrt{2} \times$ radius.

115. The maximum length cut by two circles from lines having a given direction is that which traverses a centre of similitude.

116. Find the greatest triangle inscribed in a given circle.

117. Find the greatest and least distances from a point P, $1\frac{1}{2}''$ from the centre of a circle of $\frac{5}{8}''$ radius, to the circle.

118. Find a point P on a circle such that the sum of squares of its distances from two fixed points is (i.) minimum, (ii.) maximum.

119. Plot a graph of some varying quantity—e.g. the join of a point on a circle to some fixed point, as the angle which this join subtends at the centre of the circle varies—and show that maxima and minima occur alternately.

120. If A, B are fixed points on a circle, find P on the circle so that $PA + PB$ is a maximum.

121. Find P in a straight line so that if A, B are fixed points on opposite sides of the line, $PA - PB$ is a minimum.

122. On a given base and with given sum of sides the maximum triangle is isosceles.

123. The maximum polygon of n sides of given perimeter is regular.

124. Of two regular polygons of the same perimeter, that which has the greater number of sides is greater; and a circle is the greatest figure of given perimeter.

125. Of right triangles on a given hypotenuse, the isosceles has the greatest perimeter.

126. Construct a point P on a circle the rectangle of whose perpendiculars on two fixed tangents is a maximum.

127. A tangent to a circle, centre O, meets two fixed tangents in P, Q; show that PQ and the triangle PQO are minima when the point of contact bisects the arc between the fixed tangents.

128. Find a point P in a straight line the sum of whose squares of distances from two fixed points A, B is a minimum. What does this become when AB is the straight line?

129. The minimum triangle formed with two fixed lines by a line through a fixed point is that whose side is bisected at this point.

130. Find a point **P** in a triangle the sum of whose squares of distances from the sides is a minimum. (The symmedian point, whose perps. are propl. to sides.)

131. Inscribe in a triangle the triangle of minimum perimeter. (The pedal triangle.)

132. Construct a tangent to a circle such that the product of perpendiculars on it from two fixed points on the circle is a maximum.

133. Of all triangles of given form whose sides pass through three given points the maximum is that whose perpendiculars to sides through those points are concurrent. (Use circles as in Constr. 43.)

134. If **O, A, B** are three points in order in a line, find **C** so that **OC** is (i.) arithmetic mean, (ii.) geometric mean, (iii.) harmonic mean of **OA, OB.** (If **O, C** divide **AB** harmonically, **OC** is H.M. of **OA, OB.**)

135. If a figure **ABC...** is moved in a plane into a position **A′B′C′...** so that **A′B′** is parallel to its old direction **AB,** then every side **C′D′** is parallel to its old position **CD.** (This process is called translation.)

136. Show that any figure **ABC...** can be converted into any congruent figure **A′B′C′...** of like aspect by translation and simple rotation.

137. A straight line **PQ** travels with one end **P** on a given figure, and keeps always parallel to a fixed direction; show that **Q** describes a congruent figure.

138. Construct a triangle, given the three medians **AX, BY, CZ.** (Translate **BC** to **AD, CA** to **AE,** then sides of tr. **BDE** are double medians, and **A** is its centroid.)

139. Construct a quadrilateral **ABCD,** given diagonals **AC, BD** and their angle, angle **A,** and (i.) side **BC,** (ii.) angle **C.** (Translate triangle **ABD,** to bring point **A** on **C,** and the constr. is obvious.)

140. If the end **A** of a straight line **AB** moving parallel to itself moves on a circle, then **B** describes a circle.

141. A straight line **PQ** is moved into any other position **P′Q′**; show that the angle between these is the difference of the angles, measured in the same sense, made by them with any third line **AB.**

142. Any straight line **AB** can be converted into any other **CD** by rotation and multiplication. (If **AB, CD** meet at **K,** circles **ACK, BDK** meet at centre of rotation **O.**) Interpret when **AB** $\parallel$ **CD.**

143. Any figure **ABC...** can be converted into any similar figure **A′B′C′...** of like aspect by rotation and multiplication.

144. If a rhombus is inscribed in a similar rhombus, each is a square.

145. A rhombus circumscribed by a square is a square.

146. Draw a square circumscribing a parallelogram, sides 1″, $1\frac{1}{2}$″, angle 80°. (Constr. 44, note.)

147. If **PR, QS** between opposite sides **AD, BC** and **AB, CD** of a parallelogram make an angle **POQ**=suppt. of **A,** then **PR : QS = AB : BC.**

148. Draw a rectangle, sides 2 : 3, circumscribing a parallelogram, sides 3 cm., 5 cm., angle 75°. (Constr. 44, note.)

149. A parallelogram (other than a square) cannot be circumscribed to a similar parallelogram. (In Constr. 44, if **PQRS** $|||$ **A′B′C′D′**, **S** is on circle **ROX**, and the construction gives the original parm.)

150. Arcs on the sides **AB, BC** of a parallelogram contain angles **A, B**, and meet in **O** when produced. Show that **O** is on **AC**.

151. Show that in Constr. 44 the points **O, P, D, X** are concyclic.

152. All triangles of given form inscribed in a given triangle have a common centre of rotation.

153. Draw a triangle, sides 1″, $1\frac{3}{8}$″, $1\frac{3}{4}$″. Inscribe an equilateral triangle with a vertex 1″ from one end of side $1\frac{3}{4}$″. (Use Constr. 46.)

154. Find the centre of rotation of all equilateral triangles of Ex. 153.

155. Inscribe an equilateral triangle of side 1″ in the triangle of Ex. 153. (Multiply the first eql. triangle from the centre of rotation to make its sides 1″, rotate a vertex on to one side.)

156. Inscribe a square in a parallelogram. (Construct an isosceles right triangle with vertex at diag. point, the others on sides of parm.)

157. The locus of points whose tangents to two circles have the ratio of the respective radii is a circle.

158. One end **A** of a straight line **AB** which traverses a fixed point traces a circle; plot the locus of a point on **AB**. (Piston of oscillating cylinder.)

159. One end **A** of a straight line **AB** of given length describes a circle, the other end **B** a straight line through its centre. Plot the locus of a point on **AB**. (Connecting-rod of crank and piston.)

160. Plot the locus of a point **P** on a straight line **AB** of given length moving with its ends on two perpendicular lines **OX, OY**. (Trammel.)

161. If **A, A′** are the extreme positions of **P** on **OX** in Ex. 160, and **PN** is perpendicular to **OX**, show that $PN^2 : AN \cdot A'N$ is constant, and hence (see Ch. V. p. 145, Ex. 5) that the locus is an ellipse.

162. Plot the locus of a point on a wheel as it rolls along a straight line. What is the locus of the centre?

163. Plot the envelope of a line **AB** of given length whose ends move on two fixed lines. Show that the envelope touches the fixed lines, and find the locus of its instantaneous centre.

164. Plot the envelope of a diameter fixed in a circle, as the circle rolls along a straight line. Indicate the instantaneous centre for one position of the circle and diameter.

165. Plot the envelope of a line **AB** of constant length whose ends move on two fixed circles. Discuss the case when **AB** has the length of a common tangent.

ANTIPARALLELS AND SYMMEDIANS.

Definition 37.—A line in a triangle, making with two sides the angles of the triangle opposite to these sides, is an antiparallel of the triangle to the third side.

166. The sides of the pedal triangle are antiparallels to the sides of the triangle.

167. The locus of mid points of antiparallels to a side of a triangle is a straight line through the opposite vertex (a **symmedian**).

168. Perpendiculars on the two sides of a triangle from a point on the symmedian of their vertex are proportional to the sides.

169. The three symmedians of a triangle meet in a point (the **symmedian point** or **syntroid**).

170. Tangents to the circumcircle of a triangle at the vertices are antiparallels to the sides.

171. A diameter of the circumcircle of a triangle through a vertex is perpendicular to the antiparallels of the opposite side.

172. The joins of the vertices of a triangle to the opposite vertices of the triangle formed by tangents to the circumcircle of the original triangle at its vertices, meet in the syntroid.

173. A bisector of angle of a triangle bisects the angle of the median and symmedian through its vertex.

174. An antiparallel to one side of a triangle forms a cyclic quadrilateral with the three sides.

175. Construct the syntroid of a triangle ABC.

176. Establish the following theorems:

(i.) If a triangle $A_1B_1C_1$ is similarly situated to any triangle ABC about the syntroid of ABC, its sides cut those of ABC in six concyclic points. (Tucker circle.)

(ii.) Parallels to the sides of a triangle through the syntroid cut the sides in six concyclic points. (Lemoine circle.)

(iii.) Antiparallels to the sides of a triangle through the syntroid meet the other sides in six concyclic points. (Cosen circle.)

(iv.) The lines bisecting the sides of the pedal triangle meet those of the triangle in six concyclic points. (Taylor circle.)

(v.) Show that (ii.), (iii.), (iv.) are Tucker circles.

177. The Brocard points, orthocentre, and syntroid are concyclic. (Brocard circle. The demonstration is much more difficult than that of the theorems of Ex. 176.)

(The symbol $\gtrdot$ denotes 'is the least quantity greater than.')

As elementary algebra gives no proof of the laws of operation of irrational numbers (e.g. $\sqrt{2}$, π), I propose here to establish their laws by means of decimals. I shall assume that terminating decimals, which can be reduced to fractions p/q, can be added or multiplied by the ordinary rules of arithmetic.

We may define algebraically that a **number** μ is **represented by a decimal** $a \cdot a_1 a_2 \ldots a_n \ldots$ *ad inf.*, when

$$\mu \gtrdot \text{all approximations } \mu_n\,;$$

where μ_n denotes the termng. decimal $a \cdot a_1 a_2 \ldots a_n$.

I. 'A decimal $a \cdot a_1 a_2 \ldots$ ad inf. represents one only ratio number.'

Take $\mathsf{OJ} = \mathsf{Y}$ as unit, make a scale to a units, a_1 tenths, a_2 hundredths, &c., in succession;
thus every nth approximation μ_n is represented;
also $(a+1)\mathsf{Y} >$ any $\mu_n \mathsf{Y}$ (since adding unity to the last digit a_n cannot make μ_n exceed $a+1$).

Take $\mathsf{OP}, = \mathsf{X}, \gtrdot$ all $\mu_n \mathsf{Y}$;
$\therefore\ \mathsf{X} : \mathsf{Y}, = \mu$ say, $\gtrdot$ all μ_n (Def. of $>$ or $<$ number);
i.e. the decimal $a \cdot a_1 a_2 \ldots$ represents μ.

Again, $\mu \mathsf{Y}$ is a magnitude X or OP,
and if ν is a number $< \mu$,
$\nu \mathsf{Y}$ is a magnitude OQ or $\mathsf{Z} < \mathsf{X}$ (Def. of less number);
but some scale divisions fall between Q and P; *
$\therefore\ \nu <$ some μ_n.

$\therefore$ a different decimal represents a less number ν, and
similarly " " " greater " λ;
$\therefore\ a \cdot a_1 a_2 \ldots$ represents one only number μ.

* The fundamental axiom of measure is Archimedes' axiom:

'Any magnitude A, however small, when multiplied by a large enough integer m, can be made to exceed any other B of the same kind.'

i.e. $m\mathsf{QP} > \mathsf{OJ}$, m large enough;
$\therefore\ \mathsf{QP} > \mathsf{OJ}/m > \mathsf{OJ}/10^n >$ some whole element of the scale.

Cor. We have shown incidentally how to construct a number μ from its decimal, and to construct the magnitude $\mu\mathsf{Y}$ from μ and Y. This is the general solution of the problem of finding a fourth term in a proportion.

II. 'The sum of the decimals of two numbers μ, ν is the decimal of their sum $\mu+\nu$.'

$$\begin{array}{l} a\cdot a_1a_2a_3\ldots \\ b\cdot b_1b_2b_3\ldots \\ \hline c\cdot c_1c_2c_3\ldots \end{array}$$

If the decimals of μ, ν are added, from the left, the resulting decimal $c\cdot c_1c_2\ldots$ is that of a number λ (by **I.**).

Then any less number $\kappa <$ some $\lambda_n < \lambda_p$ say, and by the process $\lambda_p < \mu_n + \nu_n$, n large enough;
$\therefore$ any less number $\kappa <$ some $(\mu_n + \nu_n)$;
but $\lambda >$ all $\mu_n + \nu_n$;
$\therefore$ $\lambda \gtrdot$ all $\mu_n + \nu_n$;
i.e. $\lambda = \mu + \nu = \nu + \mu$ (similarly).

Thus we can add and subtract numbers by their decimals.

The fact $\mu + \nu = \nu + \mu$ is a **commutative law.**

III. 'The product of the decimals of two numbers is the decimal of their product.'

$$\begin{array}{l} a\cdot a_1a_2a_3\ldots \\ b\cdot b_1b_2b_3\ldots \\ \hline c\cdot c_1c_2c_3\ldots \end{array}$$

If the decimals of μ, ν are multiplied, from the left, the resulting decimal is that of a number λ.

And any less number $\kappa < \lambda_p$ say, $< \mu_n\nu_n$, n large enough,*
and $\lambda >$ all $\mu_n\nu_n$;
$\therefore$ $\lambda \gtrdot$ all $\mu_n\nu_n$;
i.e. $\lambda = \mu\nu = \nu\mu$, similarly (**commutative law**).

Thus we can multiply and divide numbers by their decimals; and we can operate fractions of ratio numbers by the ordinary rules of arithmetic, defining the fraction λ/ν as μ, where $\mu\nu = \lambda$. (Rouse Ball's Algebra.)

Note. This can be extended to include the complete theory of indices and logarithms.

* This is a consequence of the process. See the corresponding stage in the proof of II. above.

The laws which we have established in I., II., III. cover all the operations of 'real' number, which may thus be defined as all number which can be represented by continuous decimals.

We can now establish the theorems of Ch. III., p. 66.

Alternando. (Exchange of second and third terms of a propn.)

$$\text{If } \mathsf{X} : \mathsf{Y} = \mathsf{Z} : \mathsf{W} = \mu, \text{ and } \mathsf{Y} : \mathsf{Z} = \nu;$$
$$\mathsf{X} = \mu\mathsf{Y} = \mu\nu\mathsf{Z}; \text{ and } \mathsf{Y} = \nu\mathsf{Z} = \nu\mu\mathsf{W};$$
$$\therefore \mathsf{X} : \mathsf{Z} = \mu\nu = \nu\mu = \mathsf{Y} : \mathsf{W}.$$

Inversion. (Inverting each ratio in a propn.)

$$\text{If } \mathsf{X} : \mathsf{Y} = \mathsf{Z} : \mathsf{W} = \mu, \text{ and } \mathsf{Y} : \mathsf{X} = \nu;$$
$$\mathsf{X} = \mu\mathsf{Y} = \mu\nu\mathsf{X}; \quad \therefore 1 = \mu\nu = \nu\mu;$$
$$\therefore \mathsf{W} = \nu\mu\mathsf{W} = \nu\mathsf{Z};$$
$$\text{i.e. } \mathsf{W} : \mathsf{Z} = \nu = \mathsf{Y} : \mathsf{X}.$$

Unit Theorems.

(i.) If $\mathsf{X} : \mathsf{Y} = \mathsf{Z} : \mathsf{Y}$, then $\mathsf{X} = \mathsf{Z}$.

$$\text{If } \mathsf{X} : \mathsf{Y} = \mu = \mathsf{Z} : \mathsf{Y},$$
$$\mathsf{X} \gtrless \text{all } \mu_n\mathsf{Y} = \mathsf{Z}.$$

(ii.) If $\mathsf{X} : \mathsf{Y} = \mathsf{X} : \mathsf{Z}$, then $\mathsf{Y} = \mathsf{Z}$.

By inversion, $\mathsf{Y} : \mathsf{X} = \mathsf{Z} : \mathsf{X}$, $\therefore \mathsf{Y} = \mathsf{Z}$, by (i.).

Summation.—If $\dfrac{\mathsf{X}_1}{\mathsf{Y}_1} = \dfrac{\mathsf{X}_2}{\mathsf{Y}_2} = \dfrac{\mathsf{X}_3}{\mathsf{Y}_3}$, &c., the magnitudes being all of one kind,*

then each ratio $= \dfrac{\mathsf{X}_1 + \mathsf{X}_2 + \mathsf{X}_3 + \dots}{\mathsf{Y}_1 + \mathsf{Y}_2 + \mathsf{Y}_3 + \dots}$.

For if each ratio $= \mu$,

$$\mathsf{X}_1 \gtrless \text{all } \mu_n\mathsf{Y}_1,\ \mathsf{X}_2 \gtrless \text{all } \mu_n\mathsf{Y}_2, \text{ \&c.};$$
$$\therefore \mathsf{X}_1 + \mathsf{X}_2 + \mathsf{X}_3 + \dots \gtrless \text{all } \mu_n(\mathsf{Y}_1 + \mathsf{Y}_2 + \mathsf{Y}_3 + \dots)$$
$$= \mu(\mathsf{Y}_1 + \mathsf{Y}_2 + \mathsf{Y}_3 + \dots).$$

Note. This theorem is the general form of the theorem

$$\mu\mathsf{X} + \mu\mathsf{Y} = \mu(\mathsf{X} + \mathsf{Y}).$$

Product Theorem.—If $\kappa, \lambda, \mu, \nu$ are measures of $\mathsf{X}, \mathsf{Y}, \mathsf{Z}, \mathsf{W}$ in proportion,

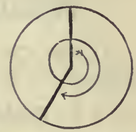
$$\kappa : \lambda = \mathsf{X} : \mathsf{Y} = \mathsf{Z} : \mathsf{W} = \mu : \nu;$$

and by ordinary algebra, $\kappa\nu = \lambda\mu$.

* Two magnitudes are of the same kind when one is equal to, greater than, or less than the other.

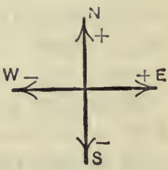
The hand of a clock may be regarded as a straight line turning in a plane, and may be said to generate the angle between two positions; in doing so it generates every possible less angle, so that an angle may be conceived as growing from zero, where the hand started, up to any given value.

And as the hand may complete 1, 2, or more turns, angles may be generated of more than 4, 8... right angles—i.e. of any magnitude whatever.



As the hand may turn either way, an angle may be diminished as well as increased; we thus get the notion of positive (+) and negative (−) angles, the positive way of turning being earthwise,* and the negative sunwise † or clockwise.

In the same way, length may be generated by a point moving along a line, lengths in opposite directions being positive and negative.



For straight lines **N** and **E** are positive, **S** and **W** negative directions.

A quantity of this type, which can be increased or diminished continuously in either sense, is **real quantity** of the most general kind. And the range of number (positive and negative) representing the measure of continuous real quantity is the whole range of **real number** of algebra. It may be defined algebraically as all number, positive and negative, that can be represented by decimals.

In the next chapter we shall assume that angles and lines have sign as well as magnitude, according to the above rules. Figures will be drawn on the assumption that the line due **E** is the zero of angular rotation; angles in any other position may be supposed to be moved so that one of their sides coincides with this position; and angles of figures will generally be considered positive. But in general theorems of angles, negative angles are included as well as positive.

* The way the earth turns round its axis, regarded from N. Pole.
† The way the sun appears to move in the sky.

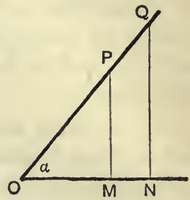
CHAPTER VI.

ELEMENTARY TRIGONOMETRY.

TRIGONOMETRICAL RATIOS—THE TRIANGLE.

If **PM**, **QN** are perpendiculars from one side of an angle α to the other,

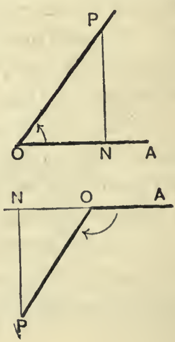
then tr. **POM** ||| **QON**;
hence the ratio **MP : OP** (= **NQ : OQ**) has a given value for each angle α; and for any greater or less angle β the ratio has a greater or less value; and similarly for ratios **OM : OP**, **MP : OM**, &c.



These ratios therefore serve to distinguish angles, though they are not directly proportional to the angles.

Definition 1.—If from a point **P** on one side of an angle **AOP** a perpendicular **PN** is drawn to the other, forming the right triangle **PON**, then the ratio

$\frac{\text{perp.}}{\text{hyp.}}$, i.e. $\frac{NP}{OP}$, is the **sine** of **AOP**;
$\frac{\text{base}}{\text{hyp.}}$, " $\frac{ON}{OP}$, " **cosine** "
$\frac{\text{perp.}}{\text{base}}$, " $\frac{NP}{ON}$, " **tangent** "
$\frac{\text{base}}{\text{perp.}}$, " $\frac{ON}{NP}$, " **cotangent** "
$\frac{\text{hyp.}}{\text{base}}$, " $\frac{OP}{ON}$, " **secant** "
$\frac{\text{hyp.}}{\text{perp.}}$, " $\frac{OP}{NP}$, " **cosecant** "



Note. In writing the ratios, always make the vertex of the angle the first, and the other end of the hypotenuse the last letter; the ratios then have always the correct sign. The last three ratios are the reciprocals of the first three, and are not much used.

The ratios are written—
sin **AOP**, cos **AOP**, tan **AOP**, cot **AOP**, sec **AOP**, cosec **AOP**;
or if a is the angle **AOP**—
sin a, cos a, tan a, cot a, sec a, cosec a.

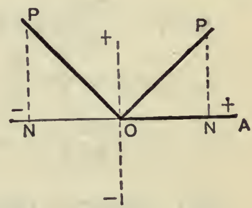
Ex. Write down all ratios for the triangle **QON**, last page.

The cosine of an angle **AOP** is the ratio of the projection (**ON**) of **OP** on **OA** to **OP** itself. Hence an important theorem—

Theorem 1.—'The projection of a straight line X on a line forming an angle a with it is X cos a.'

Algebraic Sign of the Ratios.

The projection **ON** of **OP** on **OA** is positive or negative according as it falls along **OA** or along **AO** produced; **NP** is positive for all angles up to 180°, **OP** (the hypotenuse) is always positive.



Hence (i.), when a is an **acute** angle, sin a is $+^{ve}$, cos a $+^{ve}$, tan a $+^{ve}$, &c.

(ii.) When a is **obtuse**, sin a is $+^{ve}$, cos a $-^{ve}$, tan a $-^{ve}$, &c.

For angles greater than 180°, less than 360°, **NP** falls on the other side of **OA**, and is then negative; **OP** is positive, and **ON** positive or negative as before.

Thus signs as well as magnitudes of the ratios are distinguished; but we shall not in general concern ourselves with angles greater than 180°.

An angle of any given figure—e.g. triangle or polygon—must be supposed to be moved so that one side lies along **OA**. Such angles are usually considered as positive.

Ex. 1. Find by drawing and measuring the first three trigonometrical ratios of 20°, 75°, 112°, 150°. Compare with tables.

Ex. 2. In each case of Ex. 1, find the sum of squares of sin a, cos a.

Ex. 3. In each case of Ex. 1, compare $\frac{\sin a}{\cos a}$ with tan a.

Ex. 4. Draw a triangle, $a=1{\cdot}5''$, $b=1{\cdot}9''$, $c=2{\cdot}7''$. Calculate the values of $a/\sin$ **A**, $b/\sin$ **B**, $c/\sin$ **C**, and measure the circumradius.

Ex. 5. In Ex. 4, calculate sin **A** cos **B** + cos **A** sin **B**; compare sin **C**.

The values of sines, cosines, &c. of angles can be found from tables, or by construction. The latter method is not applicable where great accuracy is required, as in ordnance survey. The ratios of angles which can be constructed by ruler and compass are easily found as surds.

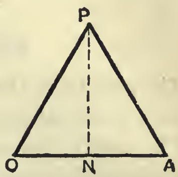
Construction 1.—(i.) '**Construct ratios of 60°, 30°.**'

If POA is an equilateral triangle, each angle is 60°; and PN, bisector of ang. P, is the rt. bisector of OA.

Hence, if each side is 2 units,

$\mathsf{ON} = 1$, $\mathsf{OP} = 2$, $\mathsf{NP}^2 = \mathsf{OP}^2 - \mathsf{ON}^2 = 3$;

$\therefore$ $\mathsf{NP} = \sqrt{3}$;



$\therefore \sin 60° = \sin \mathsf{AOP} = \dfrac{\mathsf{NP}}{\mathsf{OP}} = \dfrac{\sqrt{3}}{2}$;

$\cos 60° = \dfrac{\mathsf{ON}}{\mathsf{OP}} = \frac{1}{2}$; $\tan 60° = \dfrac{\mathsf{NP}}{\mathsf{ON}} = \sqrt{3}$.

Also, ang. $\mathsf{NPO} = \dfrac{\mathsf{P}}{2} = 30°$;

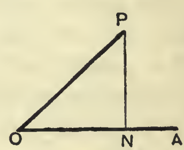
$\therefore \sin 30° = \dfrac{\mathsf{NO}}{\mathsf{PO}} = \frac{1}{2}$; $\cos 30° = \dfrac{\mathsf{PN}}{\mathsf{PO}} = \dfrac{\sqrt{3}}{2}$; $\tan 30° = \dfrac{1}{\sqrt{3}}$.

(ii.) '**Construct ratios of 45°.**'

If PON is an isosceles right triangle, ang. $\mathsf{AOP} = \mathsf{NPO} = \frac{1}{2}$ rt. ang. $= 45°$.

And if $\mathsf{ON} = 1$, $\mathsf{NP} = 1$, $\mathsf{OP}^2 = 1 + 1 = 2$;

$\therefore$ $\mathsf{OP} = \sqrt{2}$;



$\therefore \sin 45° = \dfrac{\mathsf{NP}}{\mathsf{OP}} = \dfrac{1}{\sqrt{2}}$;

$\cos 45° = \dfrac{\mathsf{ON}}{\mathsf{OP}} = \dfrac{1}{\sqrt{2}} \left(= \dfrac{\sqrt{2}}{2} \right)$; $\tan 45° = \dfrac{\mathsf{NP}}{\mathsf{ON}} = 1$.

For calculation use $\sqrt{2} = 1{\cdot}414$, $\sqrt{3} = 1{\cdot}732$.

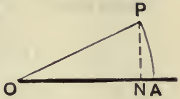
Ex. 1. Calculate the above ratios to four decimal places, and compare with the tables.

Ex. 2. Calculate the ratio of a shorter diagonal of a regular hexagon to its side.

Ex. 3. Calculate side of insquare of circle, rad. 3 cm.

Ex. 4. Calculate length of shadow of a tower 150 ft. high, when the sun's altitude (angle of elevation) is 60°.

Construction 2.—'Construct ratios of 0°, 90°, 180°.'

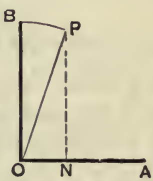


If AOP is an acute angle, PA an arc, centre O, and $PN \perp OA$;
then if P moves to coincidence with A,
the ang. AOP, arc AP, perp. NP, all become zero, and OP and ON coincide with OA.

$$\therefore \text{Sin AOP} \left(=\frac{NP}{OP}\right) \text{ becomes } \frac{0}{OA}; \text{ i.e. } \sin 0 = 0.$$

$$\text{Cos AOP} \left(=\frac{ON}{OP}\right) \quad " \quad \frac{OA}{OA}; \text{ i.e. } \cos 0 = 1.$$

$$\text{Tan AOP} \left(=\frac{NP}{ON}\right) \quad " \quad \frac{0}{OA}; \text{ i.e. } \tan 0 = 0.$$



Similarly, if $AOB = 90°$, $AOP < 90°$, PB an arc, centre O, and $PN \perp OA$;
then if P moves to coincidence with B,
ang. AOP becomes 90°, ON becomes zero,
OP and NP coincide with OB.

$$\therefore \text{Sin AOP becomes } \frac{OB}{OB}; \text{ i.e. } \sin 90° = 1.$$

$$\text{Cos AOP} \quad " \quad \frac{0}{OB}; \text{ i.e. } \cos 90° = 0.$$

$$\text{Tan AOP} \quad " \quad \frac{OB}{0}; \text{ i.e. } \tan 90° = \infty.$$

Prove similarly:

$$\text{Sin } 180° = 0, \ \cos 180° = -1, \ \tan 180° = 0.$$

Ex. 1. Find the ratios of 270° and 360°.

Ex. 2. For any triangle ABC, write down the values of

(i.) $\sin(A+B+C)$, $\cos(A+B+C)$, $\tan(A+B+C)$;

(ii.) $\sin\dfrac{A+B+C}{2}$, $\cos\dfrac{A+B+C}{2}$, $\tan\dfrac{A+B+C}{2}$.

Ex. 3. In a right triangle, A the right angle, R the circumradius, show that $2R = \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.

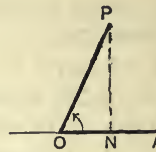
Ex. 4. Can you prove Ex. 3 for an acute triangle?

Ex. 5. If $\alpha < 90°$, show that $\tan\alpha : \sin\alpha > 1$.

Ex. 6. What does the ratio $\tan\alpha : \sin\alpha$ become when α is zero?

Ex. 7. Show that $\cos\alpha$, $\sin\alpha$ never exceed unity, but that $\tan\alpha$ can be used to represent any real number.

A few theorems follow readily from the definitions. They should be learnt by heart.

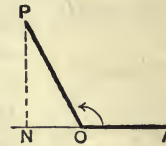


Theorem 2.—'If α is any angle'—

(i.) $\dfrac{\sin\alpha}{\cos\alpha} = \tan\alpha = \dfrac{1}{\cot\alpha}$;

(ii.) $\sin^2\alpha + \cos^2\alpha = 1$;

(iii.) $1 + \tan^2\alpha = \dfrac{1}{\cos^2\alpha} = \sec^2\alpha$.



(i.) results from writing down the ratios.

(ii.) $\text{Sin}^2\alpha + \cos^2\alpha = \dfrac{NP^2}{OP^2} + \dfrac{ON^2}{OP^2} = \dfrac{NP^2 + ON^2}{OP^2}$

$= \dfrac{OP^2}{OP^2} = 1.$

(iii.) $1 + \tan^2\alpha = 1 + \dfrac{NP^2}{ON^2} = \dfrac{ON^2 + NP^2}{ON^2} = \dfrac{OP^2}{ON^2} = \dfrac{1}{\cos^2\alpha} = \sec^2\alpha.$

Ex. 1. Prove that $\cos\alpha = \dfrac{1}{\sec\alpha}$, $\sin\alpha = \dfrac{1}{\text{cosec}\,\alpha}$, and reciprocally.

Ex. 2. Given $\sin\alpha = {\cdot}816$, calculate $\cos\alpha$, $\tan\alpha$; given $\cos\alpha = {\cdot}387$, calculate $\sin\alpha$, $\tan\alpha$; given $\tan\alpha = 1{\cdot}35$, calculate $\cos\alpha$, $\sin\alpha$.

Ex. 3. If C is a right angle in a triangle, $\sin^2 A + \sin^2 B = 1$.

When the sine, cosine, or tangent (or any ratio) of an angle is given, the other ratios may easily be found in terms of the given ratio by construction.

Construction 3.—'Determine the other ratios of an angle in terms of the given sine, cosine, or tangent.'

If $\sin\alpha$ is given $= s$, say,
make the hypotenuse OP unity; $\therefore$ perp. $NP = s$;
$\therefore$ base $ON = \sqrt{1 - s^2}$;
$\therefore$ $\cos\alpha = \sqrt{1 - s^2} = \sqrt{1 - \sin^2\alpha}$;

$\tan\alpha = \dfrac{s}{\sqrt{1 - s^2}} = \dfrac{\sin\alpha}{\sqrt{1 - \sin^2\alpha}}$, &c.

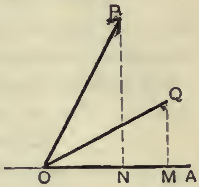
Similarly, if $\cos\alpha$ is given $= c$, say,
make hyp. OP unity; $\therefore$ base $ON = c$, &c., as before.

And if $\tan\alpha$ is given $= t$, say,
make base ON unity; $\therefore$ perp. $NP = t$;
$\therefore$ hyp. $OP = \sqrt{1 + t^2}$, &c., as before.

Ex. Given $\sin\alpha = x$, $\cos\beta = y$, $\tan\gamma = z$, find by construction the other two ratios of α, β, γ.

Theorem 3.—'**The sine of an angle = cosine of its complement;**
the cosine " = sine " " ;
the tangent " = cotangent " " .'

If AOP is α, and PN $\perp$ OA,
make AOQ $= 90° - \alpha$, and QM $\perp$ OA;
$\therefore$ tr. OPN ||| QOM;



$$\therefore \sin \alpha = \frac{NP}{OP} = \frac{OM}{OQ} = \cos (90° - \alpha);$$

Similarly, $\cos \alpha = \ldots = \ldots = \sin (90° - \alpha)$;
$\tan \alpha = \ldots = \ldots = \cot (90° - \alpha)$.

Note. These theorems are true in magnitude and sign for all angles; a similar proof, with due care as to sign, applies to all cases.

Theorem 4.—'**Sin $(90° + \alpha) = \cos \alpha$; and cos $(90° + \alpha) = -\sin \alpha$.**'

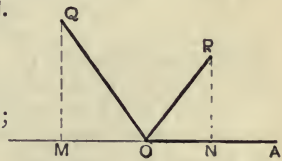
With due care as to sign, this may be proved after the same manner as Th. 3. It also is true for all angles.

Ex. 1. Fill up the following equalities, and verify from the tables: $\sin 60° = \cos\ldots$; $\sin 150° = \cos\ldots$; $\cos 54° = \sin\ldots$; $\cos 120° = -\sin\ldots$; $\tan 48° = \cot\ldots$; $\cot 37° = \tan\ldots$

Ex. 2. In any triangle $\sin \frac{C}{2} = \cos \frac{A+B}{2}$; $\cos \frac{B}{2} = \sin \frac{A+C}{2}$.

Theorem 5.—'**The sine of an angle = sine of its supplement;**
the cosine " = – cosine " ;
the tangent " = – tangent " .'

If AOP is α, and PN $\perp$ OA,
make AOQ $= 180° - \alpha$, and QM $\perp$ OA;
$\therefore$ ang. QOM = PON, and tr. OQM ||| OPN.



$$\therefore \text{Sin } \alpha = \frac{NP}{OP} = +\frac{MQ}{OQ} = \sin (180° - \alpha);$$

$$\cos \alpha = \frac{ON}{OP} = -\frac{OM}{OQ} = -\cos (180° - \alpha);$$

$$\tan \alpha = \frac{NP}{ON} = -\frac{MQ}{OM} = -\tan (180° - \alpha).$$

The theorem can be proved in a similar manner, with due care as to sign, for all values of α.

Ex. 1. By the aid of the tables, write down the numerical values of the three first ratios of 172°, 158°, 110°, 137°.

Ex. 2. Fill up the equalities, $\sin 25° = \sin\ldots$; $\cos 74° = -\cos\ldots$; $\tan 57° = -\tan\ldots$

EXAMPLES—XXXV.

(In the following examples write $\frac{1}{\sin\alpha}$, $\frac{1}{\cos\alpha}$, $\frac{1}{\tan\alpha}$ or $\frac{\cos\alpha}{\sin\alpha}$ respectively for cosec α, sec α, cot α where they occur. Also π stands for an angle of two right angles.)

1. If $\alpha=60°$, $\beta=45°$, $\gamma=30°$,

(i.) $\cos^2\alpha+3\cos\alpha-\sin^2\alpha=1$;

(ii.) $\cos\alpha\sin\gamma+\cos\beta\sin\beta+\cos\gamma\sin\alpha=1\frac{1}{2}$;

(iii.) $\dfrac{\cos\alpha+\cos\beta+\cos\gamma}{\sin\alpha+\sin\beta+\sin\gamma}=1.$

2. Find the value of (i.) $\cos 60°\sin 30°+\tan 60°\cot 30°$;

(ii.) $\sec 45°+\operatorname{cosec} 45°$;

(iii.) $\dfrac{\tan\frac{\pi}{6}+\tan\frac{\pi}{4}+\tan\frac{\pi}{3}}{\cot\frac{\pi}{6}+\cot\frac{\pi}{4}+\cot\frac{\pi}{3}}$.

3. Find the value of $\sin\alpha+\cos\alpha+\tan\alpha$, when

(i.) $\alpha=\frac{\pi}{3}$, (ii.) $\alpha=45°$, (iii.) $\alpha=\frac{\pi}{6}$.

4. Show that (i.) $\dfrac{1}{\cot\alpha+\tan\alpha}=\sin\alpha\cos\alpha$;

(ii.) $\dfrac{1-\tan\alpha}{1+\tan\alpha}=\dfrac{\cot\alpha-1}{\cot\alpha+1}$;

(iii.) $\dfrac{\tan\alpha+\tan\beta}{\cot\alpha+\cot\beta}=\tan\alpha\tan\beta.$

5. Show that (i.) $(\sin\alpha+\cos\alpha)^2=1+2\sin\alpha\cos\alpha$;

(ii.) $\sin^4\theta+\cos^4\theta=1-2\sin^2\theta\cos^2\theta$;

(iii.) $\sin^6\alpha+\cos^6\alpha=1-3\sin^2\alpha+3\sin^4\alpha$;

(iv.) $\dfrac{1+\tan^2\theta}{1+\cot^2\theta}=\tan^2\theta.$

6. Find the values of $\sin\alpha\cos\beta+\cos\alpha\sin\beta$, and compare with the values of $\sin(\alpha+\beta)$ (found from tables) in each case, when

(i.) $\alpha=30°$, $\beta=45°$; (ii.) $\alpha=30°$, $\beta=60°$; (iii.) $\alpha=60°$, $\beta=45°$.

(You may assume that $\sin 105°=\sin 75°$ for the last case.)

7. Solve the equations:

(i.) $\cos\theta=\dfrac{1}{2\cos\theta}$; (ii.) $4\sin\theta=\dfrac{3}{\sin\theta}$; (iii.) $3\tan\theta=\dfrac{1}{\tan\theta}$;

(iv.) $4\sin^3\theta=3\sin\theta$; (v.) $\sec\theta-\cos\theta=\frac{3}{2}$.

8. (i.) If $\sin\alpha=\dfrac{a}{\sqrt{a^2+b^2}}$, find $\cos\alpha$, $\tan\alpha$.

(ii.) If $\cos\alpha=\dfrac{1}{\sqrt{1+x^2}}$, find $\sin\alpha$, $\cot\alpha$.

8. (iii.) If $\tan\alpha=\dfrac{p-q}{p+q}$, find $\sin\alpha$, $\cos\alpha$.

9. In any triangle (i.) $\cos\dfrac{B+C}{2}=\sin\dfrac{A}{2}$; (ii.) $\tan\dfrac{B+C}{2}=\cot\dfrac{A}{2}$;

(iii.) $\sin(B+C)=\sin A$; (iv.) $\cos(A+B)=-\cos C$.

10. Write down from the tables the sines and cosines of 63°, 42°, 31°; and make a table showing what other angles less than 180° have the same numerical values of sine or cosine; stating in each case the algebraical sign of the ratio.

11. Find, by comparing with the corresponding acute angle, the ratios $\sin 150°$, $\cos 135°$, $\tan\dfrac{2\pi}{3}$, $\sin 225°$, $\cos\dfrac{5\pi}{3}$, $\tan 330°$.

12. If $\sin\alpha=k$, $\cos\beta=l$, find the value of

(i.) $\text{Sin}\,\alpha\cos\beta+\cos\alpha\sin\beta$.

(ii.) $\text{Cos}\,(\pi-\alpha)$, $\sin(\pi+\beta)$, $\tan(\pi-\beta)$, $\cos(\pi+\alpha)$.

(Use tables in the following examples.)

13. When the angle of elevation of the sun is 35°, a house throws its shadow just across a street 100 ft. wide. Calculate the height of the house.

14. From the top of a cliff 200 ft. high the angle of depression or dip—i.e. angle below the horizontal—of a ship is 15°. Calculate its distance from the foot of the cliff.

15. From a boat at sea the angles of elevation of the top and bottom of a lighthouse are 36° and 30°. If the cliff on which the house stands is 180 ft. high, what is the height of the house?

16. A hill has a slope of 1 in 15 (1 ft. rise in every 15 ft. along the slope). Half a mile up is a coal-shaft whose foot is on a level with that of the hill. Find the slope of the hill, and the depth in feet of the shaft.

17. From two boats at sea in line with a headland, and half a mile apart, the angles of elevation of the headland are 14° 29′ and 7° 15′ respectively. Find the height of the headland.

18. From two points 100 yd. apart on the bank of a river the directions from the river line of a post on the opposite bank are 26° 34′ and 14° 2′ respectively. Find the width of the river.

19. From a ship sailing due north for the mouth of a harbour a church spire near the shore is seen in a direction of 55° E. of N. After another mile its direction is 65°. If the coast runs E. and W., find the distance of the church from the harbour mouth.

20. A church tower 150 ft. high stands on a hill 300 ft. high. Find the angle which it subtends at a point on the flat at the foot of the hill 300 yd. distant in direct line from the foot of the tower.

Theorem 6.—'In any triangle'—

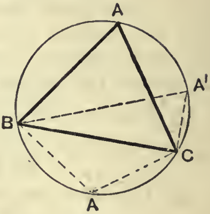
(i.) $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R.$ (Sine formula.)

(ii.) $a = b \cos C + c \cos B.$ (Projection formula.)

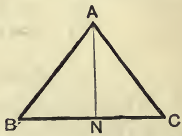
(i.) Draw the circumcle., and diam. BA';
then either ang. $A' = A$, same arc,
or ang. $A' =$ suppt. of A, opp. arc;
also ang. $BCA' = 90°$ (ang. in semcle.);

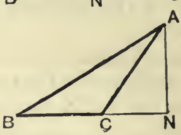
$\therefore$ in each case, $\sin A = \sin A' = \dfrac{CB}{A'B} = \dfrac{a}{2R}$;

i.e. $\dfrac{a}{\sin A} = 2R =$ (similarly) $\dfrac{b}{\sin B} = \dfrac{c}{\sin C}.$



(ii.) If $AN \perp BC$, $BC = BN + NC$, fig. (i.),
and $BC = BN - CN$, fig. (ii.).
But $NC = b \cos C$, in fig. (i.),
and $CN = b \cos ACN = b \cos (180° - C)$
$= -b \cos C$, in fig. (ii.);
and $BN = c \cos B$ in each case;
$\therefore a = BC = b \cos C + c \cos B.$





Ex. Write down the formulæ for b and c.

Ex. 1. Show that $a \sin B = b \sin A$, $b \sin C = c \sin B$.

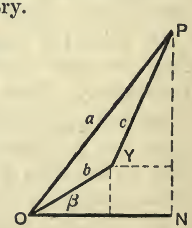
Ex. 2. Show that $(\sin A + \sin B) \div \sin C = (a + b) \div c$.

Ex. 3. Show that $a \cos B = c - b \cos A$;
$a \cos C = b - c \cos A$.

Ex. 4. Calculate R and the elements—i.e. sides and angles—of a triangle, given:
(i.) $a = 530$ yd., $A = 50°$, $B = 75°$; (ii.) $b = 12{\cdot}3$ cm., $A = 87°$, $C = 48°$.

Note. Formulæ (i.) and (ii.) are really particular cases of projection formulæ, useful in mechanics and co-ordinate geometry.

If a path b in direction β from a given line,
$+$ a path c " " γ " the " ,
are equivalent to
a path a in " α " " " ;
then $a \cos \alpha \ (= ON) = b \cos \beta + c \cos \gamma$,*
$a \sin \alpha \ (= NP) = b \sin \beta + c \sin \gamma$.
In formulæ (i.) and (ii.) $\alpha = 0$, $\beta = B$, $\gamma = -C$.



* These formulæ are true for all values of the angles α, β, γ when these are measured in the same sense.

Theorem 6.—(iii.) $a^2 = b^2 + c^2 - 2bc \cos A$; $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$. (Cosine formula.)

(iv.) $\text{Sin}\,\dfrac{A^*}{2} = \sqrt{\left\{\dfrac{(s-b)(s-c)}{bc}\right\}}$; $\cos\dfrac{A^*}{2} = \sqrt{\left\{\dfrac{s(s-a)}{bc}\right\}}$;

$\tan\dfrac{A^*}{2} = \sqrt{\left\{\dfrac{(s-b)(s-c)}{s(s-a)}\right\}}$. (Half angle formulæ.)

(iii.) From formula (i.) $a \sin B = b \sin A$;

" " (ii.) $a \cos B = c - b \cos A$.

Square and add;

$\therefore\ a^2(\sin^2 B + \cos^2 B) = b^2(\sin^2 A + \cos^2 A) + c^2 - 2bc \cos A$;

but $\sin^2 B + \cos^2 B = 1 = \sin^2 A + \cos^2 A$ (Th. 2, ii.);

$\therefore\ a^2 = b^2 + c^2 - 2bc \cos A$.

Hence, also, $2bc \cos A = b^2 + c^2 - a^2$;

i.e. $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$.

Ex. Write down the formulæ for b^2, c^2, $\cos B$, $\cos C$.

(iv.) If R, R′ are points of contact of in- and e-circles, centres I, E_1, on AB;

then $(s-b)(s-c) = r \, . \, r_a$ (Th. 100, Ch. V.),

and $bc = AI \, . \, AE_1$ " ;

$\therefore\ \sin^2\dfrac{A}{2} = \dfrac{RI}{AI} \cdot \dfrac{R'E_1}{AE_1} = \dfrac{r\,.\,r_a}{bc} = \dfrac{(s-b)(s-c)}{bc}$;

$\cos^2\dfrac{A}{2} = \dfrac{AR}{AI} \cdot \dfrac{AR'}{AE_1} = \dfrac{s(s-a)}{bc}$;

$\tan^2\dfrac{A}{2} = \dfrac{RI}{AR} \cdot \dfrac{R'E_1}{AR'} = \dfrac{(s-b)(s-c)}{s(s-a)}$.

Ex. Write down the formulæ for $\dfrac{B}{2}, \dfrac{C}{2}$.

Formula (i.) is used when a side and two angles are given; formula (iii.) or (iv.) is used when three sides are given.

For logarithms always use (iv.) and not (iii.); without logarithms use (iii.) and not (iv.).

Ex. 1. Calculate the angles, given (i.) $a = 510$ yd., $b = 750$, $c = 660$; (ii.) $a = 5$ cm., $b = 6$ cm., $c = 7$ cm.; (iii.) $a = 53{\cdot}71$ ft., $b = 68{\cdot}72$ ft., $c = 91{\cdot}34$ ft.

Ex. 2. In formula (ii.), $c = a \cos B + b \cos A$, write $2R \sin A$, &c., for a, b, c; hence prove $\sin(A + B)\ (= \sin C) = \sin A \cos B + \cos A \sin B$.

* The sign of the sq. root is +, $\because \dfrac{A}{2}$ is acute.

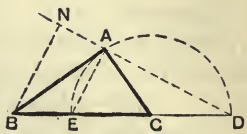
Theorem 6.—(v.) $\dfrac{\tan\frac{A-B}{2}}{\tan\frac{A+B}{2}}=\dfrac{a-b}{a+b}$.* (Tan. formula.)

(vi.) $\triangle=\sqrt{\{s(s-a)(s-b)(s-c)\}}=\frac{1}{2}bc\sin A=$&c.

(vii.) $R=\dfrac{abc}{4\triangle}=\dfrac{a}{2\sin A}$, &c.

(viii.) $r=\dfrac{\triangle}{s}=(s-a)\tan\dfrac{A}{2}$, &c.

$r_a=\dfrac{\triangle}{s-a}=s\tan\dfrac{A}{2}$, &c.



(v.) In BC make $CD=CE=CA$;

$\therefore$ C is centre of semcle. EAD;

$\therefore$ BN, perp. to AD, $\parallel$ EA.

Now ang. $D=\frac{1}{2}C$, at cent. of semcle.;

$\therefore$ $NBD=$ compt. of $D=90^\circ-\dfrac{C}{2}=\dfrac{A+B}{2}$;

and $NBA=NBD-B=\dfrac{A+B}{2}-B=\dfrac{A-B}{2}$;

$$\therefore \frac{\tan\frac{A-B}{2}}{\tan\frac{A+B}{2}}=\frac{NA}{BN}\div\frac{ND}{BN}=\frac{NA}{ND}=\frac{BE}{BD}\ (\because\ BN\parallel EA)=\frac{a-b}{a+b}.$$

Write similarly the forms for $\dfrac{A-C}{2}$, $\dfrac{C-B}{2}$.

Ex. If $a=7$, $b=5$, $C=60^\circ$; then $\tan\dfrac{A-B}{2}=2{\cdot}886$;

$\therefore \dfrac{A-B}{2}=16^\circ{\cdot}1$, and $\dfrac{A+B}{2}=60^\circ$, $\therefore$ $A=76^\circ{\cdot}1$, $B=43^\circ{\cdot}9$;

whence, by sine form., $c=6{\cdot}25$.

(vi.) The first form is Th. 100, Ch. V.; the second, $\frac{1}{2}$ base × alt.

(vii.) Th. 101, Ch. V., and 6 (i.) above.

(viii.) Th. 100, Ch. V. The second form is $\tan\dfrac{A}{2}=\dfrac{RI}{AR}=\dfrac{R'E}{AR'}$ in 6 (iv.) above.

Ex. 1. Find the elements of a triangle, given:

(i.) $b=15$, $c=12$, $A=50^\circ$; (ii.) $a=10{\cdot}5$, $c=13{\cdot}2$, $B=84^\circ{\cdot}5$.

Ex. 2. Show that $\sin B=2\sqrt{\{s(s-a)(s-b)(s-c)\}}\div ac$.

Ex. 3. Find $\triangle$, R, r, r_b, given (i.) $a=12$, $b=15$, $c=18$; (ii.) $b=25$, $B=60^\circ$, $s=36$.

* This is more convenient than the old form, $\tan\dfrac{A-B}{2}=\dfrac{a-b}{a+b}\cos\dfrac{C}{2}$.

By means of the formulæ just established, any required element (side, angle, &c.) of a triangle can be calculated, by tables, with greater accuracy than is possible by geometric construction.

Construction 4.—'Trigonometrical solution of a triangle.'

(i.) **'Two angles and one side given, A, B, c.'** (Sine F.)

Calculate $\mathsf{C} = 180° - \mathsf{A} - \mathsf{B}$.

Then $\dfrac{a}{\sin \mathsf{A}} = \dfrac{c}{\sin \mathsf{C}}$, whence a, and similarly b.

(ii.) **'Three sides given, a, b, c.'** (Half ang. F.)

Calculate $s = \dfrac{a+b+c}{2}$, $s-a$, $s-b$, $s-c$.

Then $\tan \dfrac{\mathsf{A}}{2} = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}$, whence A, and similarly B; then $\mathsf{C} = 180° - \mathsf{A} - \mathsf{B}$.

(iii.) **'Two sides and their angle given, a, b, C.'** (Tan F.)

Calculate $\dfrac{\mathsf{A}+\mathsf{B}}{2} = 90° - \dfrac{\mathsf{C}}{2}$.

Then $\tan \dfrac{\mathsf{A}-\mathsf{B}}{2} = \dfrac{a-b}{a+b} \times \tan \dfrac{\mathsf{A}+\mathsf{B}}{2}$, whence $\dfrac{\mathsf{A}-\mathsf{B}}{2}$; whence A and B by addition and subtraction.

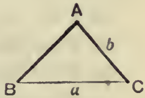
Also, $\dfrac{c}{\sin \mathsf{C}} = \dfrac{a}{\sin \mathsf{A}}$; whence c.

(iv.) The ambiguous case.

'Two sides and the angle opposite one given, a, b, A.' (Sine F.)

$\operatorname{Sin} \mathsf{B} = \dfrac{b}{a} \sin \mathsf{A}$, whence B.

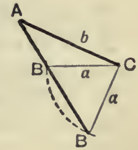
Then $\mathsf{C} = 180° - \mathsf{A} - \mathsf{B}$, and $c = a \times \dfrac{\sin \mathsf{C}}{\sin \mathsf{A}}$.



But the value of sin B does not show whether B is acute or obtuse, since sin $(180° - \mathsf{B})$ has the same value as sin B. (Th. 5.)

If, however, (i.) $b \leqq a$, then $\mathsf{B} \leqq \mathsf{A}$;

$\therefore$ B is acute, and there is only one solution; but if (ii.) $b > a$, there are two possible solutions, the two values of B being supplementary.



There are then also two values of C and of c.

(v.) **'The right triangle.'**

This reduces to the definitions of ratios.

EXAMPLES—XXXVI.

(Angles may be found to the nearest minute, sides to four figures.)

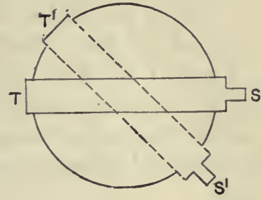
In a triangle—

1. $a=5$ ft., $b=7$ ft., $A=45°$; find B.
2. $b=21{\cdot}6''$, $c=14{\cdot}4''$, $C=34°\ 25'$; find B.
3. $a=1320$ yd., $c=1760$ yd., $C=76°\ 30'$; find A.
4. $a=7$ ft., $b=5$ ft., $c=4$ ft.; find B and C. What is A?
5. $a=22$ ft., $b=44$ ft., $c=55$ ft.; find A and C.
6. $a=3''$, $b=4''$, $C=51°\ 19'$; find c.
7. $a=10$ ft., $c=12$ ft., $B=60°$; find b.
8. $A=82°$, $B=68°$, $c=100$ yd.; find b.
9. $A=43°$, $C=77°$, $b=150$ yd.; find c.

(In the following examples four-figure logarithms should be used.)

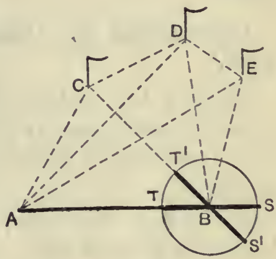
10. $a=379$, $B=86°$, $C=52°$; find b.
11. $c=4765$, $A=118°$, $B=27°$; find a.
12. $b=391{\cdot}2$, $C=102°$, $A=34°$; find c and a.
13. $a=143{\cdot}1$, $b=241{\cdot}2$, $c=157{\cdot}3$; find A.
14. $a=31{\cdot}2$, $b=21{\cdot}3$, $c=27{\cdot}9$; find B.
15. $a=415{\cdot}7$, $b=347{\cdot}9$, $c=521{\cdot}6$; find C and A.
16. $b=354{\cdot}7$, $c=426{\cdot}8$, $A=49°\ 16'$; find C.
17. $c=12{\cdot}36$, $a=21{\cdot}87$, $B=73°\ 37'$; find A.
18. $a=2087$, $b=3075$, $C=103°\ 15'$; find B and c.
19. $a=3547$, $b=4690$, $A=37°\ 45'$; find B.
20. $b=5665$, $c=6006$, $B=39°\ 30'$; find C.
21. $c=824{\cdot}3$, $a=958{\cdot}2$, $C=45°\ 43'$; find A and b.
22. Find the greatest angle of the triangle which has the least side $5''$, the sum of its sides $21''$, and the three sides in arithmetic progression.
23. Find the circumradii of the triangles in Exx. 12, 13.
24. Find the r and r_a of the triangles in Exx. 13, 14.
25. Find the areas of the triangles in Exx. 15, 16.
26. Find in terms of x and α the third side of a triangle whose sides are $x \sin \alpha$, $x \cos \alpha$, and included angle 60°. Find the value of this side when $\alpha=45°$. What kind of triangle is it in this case?
27. If $a=2b$, and $C=120°$, find the ratio $a:c$.
28. If the second greatest angle of a triangle is 45° 10′, and the sides are in geometrical progression, find (as a surd) the common ratio.
29. Calculate the other sides and angles of the triangle of Ex. 28 when the second greatest side is $1''$.
30. Solve the mean base triangle (Constr. 16, Ch. III.) when the base is 5 cm.

For purposes of mapping and survey, a **theodolite** is used to measure angles.



It consists of a telescope TS mounted on an axis so as to turn about a divided circle showing degrees and (with the aid of a vernier) minutes * of angle.

The eye-piece has a spider line by which the telescope can be accurately focussed on to a given point. With the best instruments differences of a quarter minute, or 15 seconds,* can be observed.



The circle and telescope can be adjusted horizontally by means of screws and levels; and the elements of a triangle ABC in a horizontal plane measured as follows.

A base line AB is measured as accurately as possible; the theodolite placed at one end B of this line, the telescope focussed on A, and its position on the circle noted.

It is then turned through the arc TT′ until focussed on C, and the new position T′S′ on the circle noted.

Thus the angle ABC (= SBS′) is measured by means of the circle; similarly, angles ABD, ABE, CBD, DBE to other points D, E, &c. are measured.

The instrument is then moved to A, and corresponding angles at A measured.

Thus triangle ABC is calculated from AB, angs. B, A; similarly trs. ABD, ABE, &c. are calculated. Thus CD, DE, &c. are known.

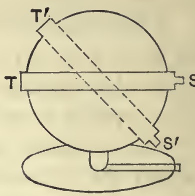
DE, for example, may now be used as a new base line, its calculated value being used. Thus a whole district or country can be mapped out very accurately by means of triangles.

This process is called **triangulation.** The calculations are made by the sine formula.

Range-finding.—This is the same process. A distance AB and angles CAB, CBA are measured, and distance AC calculated.

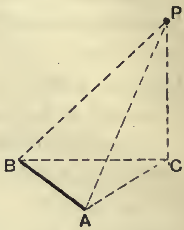
* One degree = 60 minutes, 1 minute = 60 seconds.
Thus 52° 17′ 26″ reads 52 degrees 17 min. 26 sec.

For measuring heights the telescope is mounted so as to turn about a second divided circle, perpendicular to the first, and which itself turns about the centre of the first circle.*



Thus, if the first circle is horizontal, the second is vertical, and the telescope turns in a vertical plane *about* the second circle, and in a horizontal plane *along with* the second circle.

Construction 5.—'Measure the height of an inaccessible peak.'



Measure the distance BA of two points in a horizontal plane as base line; (when AB is not horizontal a corresponding correction is required);
then if P is the peak, PC perp. to the horizontal through AB;
measure the elevation PAC at A, by the vertical circle.

Measure angle CAB by turning the vertical circle until the telescope focusses on B.

Similarly measure angle CBA.

Calculate CA, in triangle BAC, given c, A, B;
then height PC = CA tan PAC, and is thus known.

EXAMPLES—XXXVII. (Use 4-figure tables.)

1. At a point A on a straight road the corner of a house bears 58° to the left. At a point B, 200 yd. farther on, it bears 82°. Find the distance of the house from A and from the road.

2. The elevation of the top of the roof of a church is 25°, and of the top of the spire 35°, from a certain point. If the roof-top is 80 ft. from the ground, what is the height of the spire?

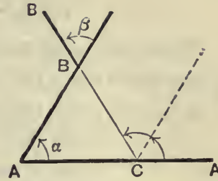
3. At a point A the peak of a mountain bears due west, and has an elevation of 22°; at a point B, 400 yd. N. from A, it bears W.S.W. If A is 2000 ft. above sea-level, find the height of the mountain.

4. A horizontal base line AB is 1235 yd. long; a mountain peak, altitude 18° 32′ at A, has directions of 82° 31′ and 88° 27′ from A, B, measured in the same sense. Find height of mountain.

* For terrestrial work only half the vertical circle is constructed.

Theorem 7.—'If α, β are any two angles,

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta;$$
$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta.\text{'}$$

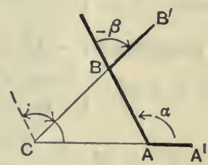


(i.) Turn a str. line AA′ through α at A into posn. AB; turn it through β at B into posn. BB′, forming tr. ABC; then

ang. $\widehat{A'CB'}=\alpha+\beta$ (by parl.ˡ to AB at C);

$$\therefore \sin(\alpha+\beta)=\sin\widehat{A'CB'}=\sin C$$
$$=\frac{c}{2R}=\frac{a\cos B+b\cos A}{2R},\text{ in tr. ABC},$$
$$=\sin A\cos B+\cos A\sin B$$
$$=\sin\alpha\cos\beta+\cos\alpha\sin\beta.$$

(ii.) Turn a str. line AA′ through α at A into posn. AB; turn it through $-\beta$ at B into posn. BB′, forming tr. ABC;



then ang. $\widehat{A'CB'}=\alpha-\beta$;

$$\therefore \sin(\alpha-\beta)=\sin\widehat{A'CB'}=\sin C$$
$$=\sin A\cos B+\cos A\sin B$$
$$=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$
$$(\text{since }\cos\alpha=-\cos A).$$

Note.* The method is readily adapted to angles of any sign and magnitude, with due care as to sign. Thus, in fig. (i.), $\alpha>270°$, $\beta>90°$;

ang. $\widehat{A'CB'}=\alpha+\beta=360°+C$;

$\therefore \sin(\alpha+\beta)=\sin(360°+C)=\sin C$
$=\sin A\cos B+\cos A\sin B.$

And $\sin\alpha=-\sin A$, $\cos\alpha=\cos A$,
$\sin\beta=\sin B$, $\cos\beta=-\cos B$;

$\therefore \sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta.$

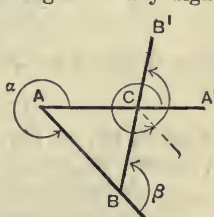


Fig. i.

In fig. (ii.), $\alpha<90°$, $\beta>180°$;

ang. $\widehat{A'CB'}=\alpha-\beta=C-180°$ ($\because \beta>\alpha$);

$\therefore \sin(\alpha-\beta)=-\sin C$
$=-\sin A\cos B-\cos A\sin B.$

And $\sin\alpha=\sin A$, $\cos\alpha=-\cos A$,
$\sin\beta=-\sin B$, $\cos\beta=-\cos B$;

$\therefore \sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta.$

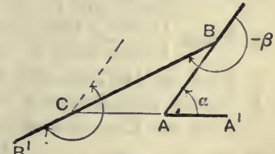


Fig. ii.

* Beginners may postpone the rest of this page.

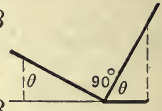
Theorem 8.—'**If** α, β **are any two angles,**

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta;$$
$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta.\text{'}$$

These are most easily derived from the sine forms.

Since $\cos\theta=\sin(90°+\theta)$, whatever ang. θ is, (Th. 4);

$$\begin{aligned}\text{Cos}(\alpha+\beta)&=\sin(90°+\alpha+\beta)\\&=\sin(90°+\alpha)\cos\beta+\cos(90°+\alpha)\sin\beta\\&=\cos\alpha\cos\beta-\sin\alpha\sin\beta.\end{aligned}$$



$$\begin{aligned}\text{Cos}(\alpha-\beta)&=\sin(90°+\alpha-\beta)\\&=\sin(90°+\alpha)\cos\beta-\cos(90°+\alpha)\sin\beta\\&=\cos\alpha\cos\beta+\sin\alpha\sin\beta.\end{aligned}$$

These results may be obtained geometrically by using $(90°+\alpha)$ instead of α in the process of Th. 7.

Note. The forms for $\sin(\alpha-\beta)$, $\cos(\alpha-\beta)$ may be derived from those of $\sin(\alpha+\beta)$, $\cos(\alpha+\beta)$ by changing the sign of β, since the original process for $\sin(\alpha+\beta)$ is quite general.

Theorem 9.—'**If** α, β **are any two angles,**

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta};\ \tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}.\text{'}$$

For $$\tan(\alpha+\beta)=\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}=\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}$$
$$=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

(dividing each term by $\cos\alpha\cos\beta$).

Prove similarly for $\tan(\alpha-\beta)$.

EXAMPLES—XXXVIII.

1. Find sin 15°, cos 105°, sin 195°, cos (−15°) as surds. (Use 60° − 45°, 60° + 45°, 45° + 150°, 45° − 60°.)
2. Prove $\dfrac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{\cos(\alpha+\beta)+\cos(\alpha-\beta)}=\tan\alpha$.
3. Prove $\dfrac{\tan\alpha+\tan\beta}{\tan\alpha-\tan\beta}=\dfrac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}$.
4. Prove $\dfrac{1+\tan\alpha\tan\beta}{1-\tan\alpha\tan\beta}=\dfrac{\cos(\alpha-\beta)}{\cos(\alpha+\beta)}$.
5. Find tan 165°, tan 105° as surds.
6. In any triangle $\tan A+\tan B+\tan C=\tan A\tan B\tan C$. (Use $\tan C=-\tan(A+B)$.)
7. Prove $\tan(45°+A)=-\dfrac{\sin A+\cos A}{\sin A-\cos A}$.

Theorem 10.—'If α, β are any two angles,

$$\sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin\alpha\cos\beta;$$
$$\sin(\alpha+\beta)-\sin(\alpha-\beta)=2\cos\alpha\sin\beta;$$
$$\cos(\alpha+\beta)+\cos(\alpha-\beta)=2\cos\alpha\cos\beta;$$
$$\cos(\alpha+\beta)-\cos(\alpha-\beta)=-2\sin\alpha\sin\beta.\text{'*}$$

These result at once from the forms of Thh. 7 and 8.

Also, by multiplication, we obtain:

$$\begin{aligned}\text{Sin}(\alpha+\beta)\sin(\alpha-\beta)&\\ &=(\sin\alpha\cos\beta+\cos\alpha\sin\beta)(\sin\alpha\cos\beta-\cos\alpha\sin\beta)\\ &=\sin^2\alpha\cos^2\beta-\cos^2\alpha\sin^2\beta\\ &=\sin^2\alpha(1-\sin^2\beta)-(1-\sin^2\alpha)\sin^2\beta\\ &=\sin^2\alpha-\sin^2\beta.\end{aligned}$$

$\text{Cos}(\alpha+\beta)\cos(\alpha-\beta)=\cos^2\alpha-\sin^2\beta$ (similarly).

Theorem 11.—'If α, β are any two angles,

$$\sin\alpha+\sin\beta=2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2};$$
$$\sin\alpha-\sin\beta=2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2};$$
$$\cos\alpha+\cos\beta=2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2};$$
$$\cos\alpha-\cos\beta=-2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}.\text{'*}$$

Make $\gamma=\dfrac{\alpha+\beta}{2}$, $\delta=\dfrac{\alpha-\beta}{2}$; $\therefore\ \gamma+\delta=\alpha$, and $\gamma-\delta=\beta$;

$$\begin{aligned}\therefore\ \sin\alpha+\sin\beta&=\sin(\gamma+\delta)+\sin(\gamma-\delta)\\ &=2\sin\gamma\cos\delta\ \text{(Th. 10)}\\ &=2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}.\end{aligned}$$

Similarly the other forms may be derived.

EXAMPLES—XXXIX.

1. Express $\sin 60°+\sin 72°$ as a product; and $\sin 60°\sin 72°$ as a difference of ratios.
2. If $2\cos 54°\cos x=\cos 150°+\cos 42°$, find x.
3. Prove $\dfrac{\sin\alpha+\sin\beta}{\cos\alpha+\cos\beta}=\tan\dfrac{\alpha+\beta}{2}$.
4. Prove $\dfrac{\cos\alpha-\cos\beta}{\sin\alpha+\sin\beta}=-\tan\dfrac{\alpha-\beta}{2}$.
5. Prove $\sin^2 75°-\sin^2 37°=\sin 112°\sin 38°$.

* It is important to remember the minus sign.

Theorem 12.—'If α is any angle,

$$\sin 2\alpha = 2\sin\alpha\cos\alpha; \qquad \sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2};$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha \qquad \cos\alpha = \cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}$$

$$= 2\cos^2\alpha - 1 \qquad = 2\cos^2\frac{\alpha}{2} - 1$$

$$= 1 - 2\sin^2\alpha; \qquad = 1 - 2\sin^2\frac{\alpha}{2};$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}; \qquad \tan\alpha = \frac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}}.$$

Make $\beta = \frac{\alpha}{2}$, $\therefore\ \alpha = 2\beta$. Then,

(i.) $\text{Sin } 2\alpha = \sin(\alpha+\alpha) = \sin\alpha\cos\alpha + \cos\alpha\sin\alpha$
$= 2\sin\alpha\cos\alpha;$
and $\sin\alpha = \sin 2\beta = 2\sin\beta\cos\beta$
$= 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}.$

(ii.) $\text{Cos } 2\alpha = \cos(\alpha+\alpha) = \cos\alpha\cos\alpha - \sin\alpha\sin\alpha$
$= \cos^2\alpha - \sin^2\alpha$
$= \cos^2\alpha - (1-\cos^2\alpha)$
$= 2\cos^2\alpha - 1;$
or $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 1 - \sin^2\alpha - \sin^2\alpha$
$= 1 - 2\sin^2\alpha;$
and $\cos\alpha = \cos 2\beta = \cos^2\beta - \sin^2\beta$
$= \cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2} = 2\cos^2\frac{\alpha}{2} - 1 = 1 - 2\sin^2\frac{\alpha}{2}.$

(iii.) $$\text{Tan } 2\alpha = \tan(\alpha+\alpha) = \frac{\tan\alpha + \tan\alpha}{1-\tan\alpha\tan\alpha} = \frac{2\tan\alpha}{1-\tan^2\alpha};$$

$$\tan\alpha = \tan 2\beta = \frac{2\tan\beta}{1-\tan^2\beta} = \frac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}}.$$

Ex. 1. If $\sin\frac{\alpha}{2} = \frac{3}{5}$, find $\cos\alpha$, $\sin\alpha$; if $\cos\frac{\theta}{2} = \frac{5}{7}$, find $\cos\theta$, $\tan\theta$.

Ex. 2. Prove $\sin 3\theta + \sin\theta = 4(\sin\theta - \sin^3\theta)$.

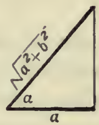
Ex. 3. Prove $\tan\frac{\alpha}{2} = \frac{\sin\alpha}{1+\cos\alpha}$.

The solution of equations involving trigonometrical ratios is of great importance in mechanics. It is effected by reduction to a known type, as a quadratic; by reduction to factors; and in the case of simultaneous equations by elimination, often a very difficult process. We give a few examples.

Construction 6.—(i.) Solve the equation $a \cos \theta + b \sin \theta = c$.

Calculate α, where $\tan \alpha = b/a$, and $r = \sqrt{a^2+b^2}$.

Then $a = r \cos \alpha$, $b = r \sin \alpha$;



$\therefore \cos(\alpha - \theta) = \cos \alpha \cos \theta + \sin \alpha \sin \theta = \dfrac{c}{r}$.

Find $\alpha - \theta$ from table, and subtract from α;

$\therefore \theta = \alpha - (\alpha - \theta)$.

This may also be done as a quadratic:

$$a\left(\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2}\right) + 2b \cos \frac{\theta}{2} \sin \frac{\theta}{2} = c\left(\sin^2 \frac{\theta}{2} + \cos^2 \frac{\theta}{2}\right);$$

$$\therefore (a+c) \sin^2 \frac{\theta}{2} - 2b \sin \frac{\theta}{2} \cos \frac{\theta}{2} + (c-a) \cos^2 \frac{\theta}{2} = 0;$$

dividing by $\cos^2 \dfrac{\theta}{2}$ gives a quadratic in $\tan \dfrac{\theta}{2}$,

$$(a+c) \tan^2 \frac{\theta}{2} - 2b \tan \frac{\theta}{2} + c - a = 0.$$

(ii.) Solve the equation $\sin 5\theta + \sin 3\theta = \cos \theta$.

Then $2 \sin 4\theta . \cos \theta = \cos \theta$;

$\therefore$ either $\cos \theta = 0$, $\theta = 90°$; or $\sin 4\theta = \dfrac{1}{2} = \sin 30°$ or $\sin 150°$,

$$\theta = 7° \, 30' \text{ or } 37° \, 30'.$$

(iii.) Solve the equations $\sin(\theta + \phi) + \sin(\theta - \phi) = a$,
$$b \cos \phi + c \sin \theta = d.$$

Then $2 \sin \theta . \cos \phi = a$, and $\cos \phi = \dfrac{d - c \sin \theta}{b}$;

$\therefore 2 \sin \theta (d - c \sin \theta) = ab$, a quadratic in $\sin \theta$.

EXAMPLES—XL.

Solve the equations:

1. $2 \cos^2\theta - 5 \cos \theta - 3 = 0$.
2. $\text{Sin } \theta + \dfrac{1}{\sin \theta} = 2\frac{1}{2}$.
3. $\text{Sin } 6\theta - \sin 4\theta = \dfrac{3}{2} \sin \theta$.
4. $2 \cos \theta - \dfrac{1}{2 \cos \theta} + 1 = 0$.
5. $\text{Sin } \theta = 2 \cos \theta$.
6. $\text{Sin } \theta = 4 \cos \theta - 1$.

It is sometimes convenient to speak of an angle in terms of its sine, cosine, and tangent. For this purpose inverse functions, $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, &c., are used. These mean

'The angle whose sine, cosine, or tangent, &c., is x.'

Ex. The slope of a hill is $\tan^{-1}\frac{1}{30}$. How much does it rise in 10 miles, measured up the hill?

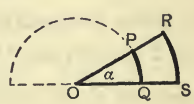
Circular Measure.

The general, as distinct from the merely geometrical, theory of trigonometrical ratios is an important part of the general theory of number. For this part of the subject the most convenient measure of angle is the ratio of two lines, since an angle may then be treated as the same kind of magnitude as its sine, cosine, or tangent.

Definition 2.—The **circular measure** of an angle is the ratio of the intercepted arc of a circle whose centre is its vertex, to the radius of that circle.

If PQ, RS are the arcs of two such circles for an angle α, and r, r' their radii,

and ρ the measure of two right angles;



then $\frac{\alpha}{\rho}=\frac{\text{PQ}}{\text{semicircle } r}=\frac{\text{RS}}{\text{semicircle } r'}$;

$\therefore\ \alpha=\frac{\rho}{\pi}\times\frac{\text{arc}}{\text{radius}}$ in each case.

Hence, by making π the measure of two right angles, the measure of α is $\frac{\text{arc}}{\text{radius}}$, which is the circular measure of α.

Angles in circular measure are generally expressed in terms of π; thus π, $\frac{\pi}{2}$, $\frac{\pi}{3}$ are 2, 1, $\frac{2}{3}$ right angles.

The actual unit is a **radian**, and is equal to $\frac{2}{\pi}$ right angles.

(For π radians = 2 right angles, $\therefore$ 1 radn. = $\frac{2}{\pi}$ rt. angs.)

Ex. The sun's disc subtends at the earth an angle of 32′, and its distance from the earth is 91,000,000 miles. Calculate its diameter. (Treat the sun's diameter as an arc of a circle, rad. 91,000,000 mi.)

π, the circular measure of two right angles, is a very convenient symbol for that angle; but it must be remembered that in such expressions as $\pi+a$, a must be expressed in *circular measure* and not in degrees. It is easy to convert angles from one system of measure to the other by the following principle:

'An angle is the same fraction* of two right angles in all systems of measurement.'

Hence, if d is the number of degrees, c of radians—i.e. the circular measure—of an angle,

$$\frac{d}{180}=\frac{c}{\pi}.$$

Ex. (i.). Express 77° in circular measure (as fraction of π).

If c is the circular measure,

$$c=\pi\times\frac{77}{180}.$$

Ex. (ii.). Express in degrees the unit of circular measure.

Here $c=1$.

$$\therefore \text{ unit of c.m.}=\frac{180^\circ}{\pi}=180^\circ\div 3{\cdot}141592...$$
$$=57^\circ{\cdot}295779...$$
$$=57^\circ\ 17'\ 45''.$$

EXAMPLES—XLI.

1. Express in circular measure (fractions of π) 60°, 75°, 108°, 135°.

2. Express in degrees $\frac{2\pi}{3}$; $\frac{\pi}{4}$; $\frac{4\pi}{5}$; $\frac{3\pi}{10}$.

3. Find the length of a degree of longitude in latitude 60°. (Earth's rad. 3960 mi.)

SINE CURVE—GRAPHS OF RATIOS.

The graph of sin x ($y=\sin x$), when x is the circular measure of the angle, is the **sine curve**; corresponding graphs give the **cosine curve** and the **tan. curve.**

If graphs of sin x, cos x, tan x are made on tenth-inch squared paper, one inch denoting unit number ($\sin\frac{\pi}{2}$ or sin 90°); then if one tenth-inch denotes 6°, 1·5″ denotes 90°; and in the true sine, &c., curves, 1·57″... denotes $\frac{\pi}{2}$, the difference being less than 5 per cent. These graphs, therefore, represent very nearly the true sine, &c., curves.

* Used in the general algebraical sense.

EXAMPLES—XLII.

TRIGONOMETRY.

1. In a right triangle, if $A=90°$ and $a=1$, the measures of sin B, sin C are b, c. What are the measures of cos B, cos C?

2. By Ex. 1 give a construction for measuring the sine and cosine of a given angle.

3. In a right triangle, if $A=90°$ and $c=1$, the measure of tan B is b. What is the measure of cot C?

4. By Ex. 3 give a construction for measuring the tangent of a given angle.

5. Construct angles of 32°, 49°, 127°, 152°, and tabulate their ratios by measuring. (Make the hyp. the unit for sines and cosines, and the base the unit for tangents.)

6. If $A=90°$ in a triangle ABC, show that $b=a\cos C=a\sin B$; and $b=c\tan B$.

7. If a is a chord of a circle, radius r, show that $a/2r$ is the sine of the angle in the larger arc of a.

8. Find by trigonometry the radius of the circumcircle of a regular pentagon of 1″ side.

9. Find the radius of the incircle of a regular hexagon of 3 cm. side.

10. Find the radii of circum- and in-circles of a regular polygon of n sides, each side a.

11. From the triangle whose base is the mean part of the other sides derive sin 18°, and compare with the table.

12. How far must the side BC, 2 ft. long, of a square ABCD be produced to P to make the angle BAP 60°?

13. Calculate the chord of a sector of a circle of radius r, angle α. What is the distance of the chord from the centre?

14. Show that the cosine and sine of an angle can have any numerical value between -1 and $+1$, but none >1 or <-1.

15. Show that any real number $\pm\mu$ may be represented by the tangent of some angle.

16. Given that $\sin\alpha=\cdot 73$, $\cos\beta=\cdot 58$, $\tan\gamma=1\cdot 62$, construct the angles α, β, γ. Measure them.

17. In a triangle ABC, incentre I, show that inradius $r=\mathrm{AI}\,.\sin\frac{A}{2}$, and $\tan\frac{A}{2}=\frac{r}{s-a}$.

18. Given the sine of an angle, e.g. $\sin\alpha=s$, show how to construct $\cos\alpha$, $\tan\alpha$. How can these be calculated without construction?

19. Given $\tan\alpha=3\cdot 5$, calculate $\sin\alpha$, $\cos\alpha$.

20. Given $\cos\alpha = \frac{2}{3}$, calculate $\sin\alpha$, $\tan\alpha$.

21. Show that $\cos^2\alpha = 1 - \sin^2\alpha$, and $\sin^2\alpha = 1 - \cos^2\alpha$.

22. Show that $\sin^3\alpha + \cos^3\alpha = (\sin\alpha + \cos\alpha)(1 - \sin\alpha\cos\alpha)$. Write down the corresponding form for $\sin^3\alpha - \cos^3\alpha$.

23. Show that $1 + 2\sin\alpha\cos\alpha = (\sin\alpha + \cos\alpha)^2$. Deduce a value for $1 - 2\sin\alpha\cos\alpha$.

24. Show that $(\sec\alpha - \tan\alpha)^2 = \frac{1 - \sin\alpha}{1 + \sin\alpha}$.

25. Trace the changes in magnitude of $\cos\alpha$, $\sin\alpha$, $\tan\alpha$, as α increases from 0° to 90°.

26. Write the following in a line: sin 177°, cos 228°, tan 296°; underneath write the sign of each; underneath again write the acute angle whose sine, cosine, tangent respectively has the same numerical value as these.

27. By the aid of the tables (which are given up to 90°), and your knowledge of the signs, write down the values of sin 250°, cos 140°, tan 260°.

28. Solve, with the aid of tables, the equations

$$7\sin\theta = 3{\cdot}06,\quad 9\cos\theta = 6{\cdot}02,\quad 5\tan\theta = 1{\cdot}457.$$

29. Solve (i.) $3\sin^2\theta - 4\sin\theta + 2 = \cos^2\theta$;
(ii.) $\sin^2\theta = 1 - 2\cos^3\theta$;
(iii.) $\tan^2\theta + \cot^2\theta = 3\frac{1}{3}$.

30. Trace the changes in magnitude and sign of $\sin\alpha$, as α increases from 0° to 180°.

31. Repeat Ex. 30, for $\cos\alpha$ and $\tan\alpha$ successively instead of $\sin\alpha$.

32. Show that $\sin(-\alpha) = -\sin\alpha$; $\cos(-\alpha) = \cos\alpha$; and $\tan(-\alpha) = -\tan\alpha$, where α is an acute angle.

33. If α is some angle $\begin{smallmatrix}>45^\circ\\<90^\circ\end{smallmatrix}$, show that there is some angle $\beta < 45^\circ$ such that $\cos\beta = \sin\alpha$.

34. How could you from a table of sines find the values of cos 38°, cos 47°, cos 75°?

35. Show that if tables of sines and cosines are both printed for all angles up to 45°, the sines and cosines of all angles up to 90° can be found from them.

36. Prove $\cos\alpha = \sin(90° - \alpha)$ when $\alpha \begin{smallmatrix}>90^\circ\\<180^\circ\end{smallmatrix}$;

and $\sin\alpha = \cos(90° - \alpha)$ when $\alpha \begin{smallmatrix}>180^\circ\\<270^\circ\end{smallmatrix}$.

37. Prove $\left.\begin{aligned}\cos\alpha &= \sin(90° + \alpha)\\ \sin\alpha &= \cos(90° + \alpha)\end{aligned}\right\}$ when $\alpha \begin{smallmatrix}>90^\circ\\<180^\circ\end{smallmatrix}$.

38. Prove $\left.\begin{aligned}\cos\alpha &= -\cos(180° - \alpha)\\ \sin\alpha &= \sin(180° - \alpha)\end{aligned}\right\}$ when $\alpha \begin{smallmatrix}>180^\circ\\<270^\circ\end{smallmatrix}$.

39. Find the sun's altitude—i.e. angle of elevation—when a post 18 ft. high throws a shadow 10 ft. long.

40. A man 6 ft. high standing 8 ft. from the base of a lamp-post has a shadow of 10 ft.; find the height of the post, and the angle it subtends at the end of the shadow.

41. An iceberg has an elevation of 15° from a ship. At $\frac{1}{20}$ mile nearer the elevation is $43\frac{1}{2}$°. What is its height?

42. If the earth's radius is 3960 miles, calculate the radius and circumference of the parallel of latitude 60°. How much faster is a point on the equator moving than a point in this latitude?

43. At a point on one diagonal of a square fort, 73·2 yd. from the nearest corner, the other diagonal subtends an angle of 60°. Find the dimensions of the fort.

44. Find the angle which a chord of 1″ subtends at the circumference of a circle of ·75″ radius. (Draw a diameter.)

45. A trapezium ABCD has angles A, B right angles, and angle ABD equal to BCD; if also AD . BC = AB . CD, find this angle.

46. Verify your result in Ex. 45 by making AB 1″, and measuring the other sides.

(In Exx. 47 to 57, a, b, c, A, B, C, &c. are sides, angles, &c. of a triangle ABC.)

47. $(b+c)\cos A+(c+a)\cos B+(a+b)\cos C=a+b+c$.

48. $c(\sin A+p\sin B)=(a+pb)\sin C$.

49. $(b+c)(1-\cos A)=a(\cos B+\cos C)$.

50. If $2s=2{\cdot}75''$, $a=1''$, $A=50°$, calculate r, R, $\triangle$.

51. If $AD \perp BC$, calculate AD, CD, given $a=1{\cdot}3''$, $b=1{\cdot}7''$, $B=78°$.

52. $(c-b\cos A)\tan B=b\sin A$.

53. $\tan A:\tan B=(c^2+a^2-b^2):(b^2+c^2-a^2)$.

54. Given $a=157$, $b=215$, $c=193$, find the greatest angle.

55. Given $A=51°\ 36'$, $B=79°\ 23'$, $c=200$ yd., find the other sides.

56. Given $a=1570$ m., $b=2396$ m., $C=111°$, find c and $\triangle$.

57. Given $B=51°\ 51'$, $b=213$ yd., $c=250$ yd., find C and a.

58. In a quadrilateral ABCD, AB is a measured base line of 1000 yd., angs. DAC, CAB = 15° 18′, 46° 42′, and angs. CBD, DBA = 21° 12′, 36° 30′; calculate the sides and angles of the quadrilateral.

59. If two men start from the corner of two straight roads to walk along them at 5 and $5\frac{1}{2}$ miles an hour respectively, and are 30 miles apart at the end of four hours, find the angle of the roads.

60. From the end A of a horizontal base line AB, 1000 m. long, the altitude of a peak P is 21° 42′; and the angles A, B which AB makes with the horizontal directions of P at these points are 87° 24′ and 72° 18′. Find the height of the peak.

61. (i.) $\text{Sin}\,(\alpha+\beta)\cos\alpha-\cos(\alpha+\beta)\sin\alpha=\sin\beta.$

(ii.) $\text{Tan}\,\frac{\alpha+\beta}{2}\tan\frac{\alpha-\beta}{2}=-\frac{\cos\alpha-\cos\beta}{\cos\alpha+\cos\beta}.$

62. (i.) $\text{Sin}\,3\alpha=3\sin\alpha-4\sin^3\alpha.$

(ii.) $\text{Cos}\,3\alpha=4\cos^3\alpha-3\cos\alpha.$

63. Express $\sin(\alpha+\beta+\gamma)$, $\cos(\alpha+\beta+\gamma)$ in terms of sines and cosines of α, β, γ; and find their values (i.) when $\alpha+\beta+\gamma=180°$, (ii.) when $\alpha=\beta=\gamma$.

64. (i.) $\text{Cos}\,(n+1)\theta\cos\theta+\sin(n+1)\theta\sin\theta=\cos n\theta.$

(ii.) $\text{Sin}\,5\alpha=16\sin^5\alpha-20\sin^3\alpha+5\sin\alpha.$

65. In any triangle $\tan A=\frac{\tan B+\tan C}{\tan B\tan C-1}$; $\tan\frac{B}{2}=\frac{1-\tan\frac{C}{2}\tan\frac{A}{2}}{\tan\frac{C}{2}+\tan\frac{A}{2}}.$

66. Simplify $\frac{\sin 3\theta+\sin 5\theta}{\cos 3\theta-\cos 5\theta}$; $\frac{\cos 2\alpha-\cos 2\beta}{\sin 2\alpha+\sin 2\beta}.$

67. Prove $1+\sin\theta=\left(\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)^2$; write down the corresponding value of $1-\sin\theta$.

68. Show that with proper signs to the roots,

$$2\sin\frac{\theta}{2}=\sqrt{(1+\sin\theta)}+\sqrt{(1-\sin\theta)}.$$

69. Simplify $\frac{\cos\alpha+\sin\alpha}{\cos\alpha-\sin\alpha}-\frac{\cos\alpha-\sin\alpha}{\cos\alpha+\sin\alpha}.$

70. Simplify $\frac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta}+\frac{\sin(\beta-\gamma)}{\cos\beta\cos\gamma}+\frac{\sin(\gamma-\alpha)}{\cos\gamma\cos\alpha}.$

71. In any triangle,

(i.) $\frac{\sin B-\sin C}{\cos B-\cos C}=-\tan\frac{A}{2}$;

(ii.) $c^2=(a-b)^2\cos^2\frac{C}{2}+(a+b)^2\sin^2\frac{C}{2}.$

72. In any triangle,

(i.) $\frac{a}{b-c}=\frac{\cos\frac{A}{2}}{\sin\frac{B-C}{2}}$;

(ii.) $\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C.$

73. A man sees a balloon due north at an elevation of 60°. Another man half a mile to the west sees it N.N.E. What is its height and its angle of elevation at the second point?

74. A house and a railway station are on opposite sides of a rectangular wood, sides 1 mile and 2 miles. Their directions from the nearest corners of the wood are parallel, at 108° to the short sides of the rectangle; and their distances from these corners are $\frac{1}{2}$ mile and $\frac{3}{4}$ mile. Find the shortest road that can be made from house to station without going through the wood.

75. Solve the equations:

(i.) $\cos 2\theta = \cos^2\theta$;
(ii.) $\tan\theta \,.\, \tan 2\theta = 1$;
(iii.) $2\cos\theta\cos 2\theta - \cos 3\theta = 1$;
(iv.) $1 - \sin^3\theta = \cos^3\theta - 2\sin\theta\cos\theta$.

76. If $\sin\alpha = \frac{2}{3}$, $\cos\beta = \frac{3}{4}$, find the values of $\sin(\alpha+\beta)$ and $\cos(\alpha-\beta)$.

77. (i.) $(1 + \cot\theta + \tan\theta)(\sin\theta - \cos\theta) = \dfrac{\sin^2\theta}{\cos\theta} - \dfrac{\cos^2\theta}{\sin\theta}$.

(ii.) $\operatorname{Cos} 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$.

78. (i.) $\operatorname{Sin}(\alpha-\beta) + \sin(\beta-\gamma) + \sin(\gamma-\alpha) = -4\sin\dfrac{\alpha-\beta}{2} \,.\, \sin\dfrac{\beta-\gamma}{2}\sin\dfrac{\gamma-\alpha}{2}$.

(ii.) $\operatorname{Sin}(\alpha+\beta+\gamma)\sin\alpha = \sin(\alpha+\beta)\sin(\alpha+\gamma) - \sin\beta\sin\gamma$.

79. Prove $\tan(\alpha+\beta+\gamma) = \dfrac{1 - \tan\alpha \,.\, \tan\beta - \tan\beta\tan\gamma - \tan\gamma\tan\alpha}{\tan\alpha + \tan\beta + \tan\gamma - \tan\alpha \,.\, \tan\beta \,.\, \tan\gamma}$;
and deduce a value for $\tan 3\alpha$ in terms of $\tan\alpha$.

80. (i.) $\dfrac{\operatorname{Sin}\alpha + 2\sin 3\alpha + \sin 5\alpha}{\operatorname{Sin} 3\alpha + 2\sin 5\alpha + \sin 7\alpha} = \dfrac{\sin 3\alpha}{\sin 5\alpha}$.

(ii.) $\operatorname{Tan}\alpha + \tan(60^\circ + \alpha) + \tan(120^\circ + \alpha) = 3\tan 3\alpha$.

81. $\{\operatorname{Sin}(\alpha+\beta) + \cos(\alpha-\beta)\}\{\cos(\alpha+\beta) + \sin(\alpha-\beta)\} = \cos 2\beta(\cos\alpha + \sin\alpha)^2$.

82. If $\alpha+\beta+\gamma = 180^\circ$, prove the following equalities:

(i.) $\sin\alpha + \sin\beta + \sin\gamma = 4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$;

(ii.) $\sin\alpha + \sin\beta - \sin\gamma = 4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\gamma}{2}$;

(iii.) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = -4\cos\alpha\cos\beta\cos\gamma - 1$;

(iv.) $\cos\frac{\alpha}{2} + \cos\frac{\beta}{2} + \cos\frac{\gamma}{2} = 4\cos\frac{\alpha+\beta}{4}\cos\frac{\beta+\gamma}{4}\cos\frac{\gamma+\alpha}{4}$;

(v.) $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2 + 2\cos\alpha\cos\beta\cos\gamma$.

83. If $\sin\alpha = \frac{4}{5}$, $\sin\beta = \frac{12}{13}$, $\sin\gamma = \frac{24}{25}$, and the angles are all acute, find the value of $\cos(\alpha+\beta+\gamma)$ as a fraction. (See Ex. 63.)

84. If $\sin\theta + \sin^2\theta = 1$, find $\sin\theta$, and show that $\cos^2\theta + \cos^4\theta = 1$.

85. If $\sin\alpha + \cos\alpha = p$, and $\tan\alpha + \cot\alpha = q$, show that $q(p^2 - 1) = 2$.

86. If $\tan^2\alpha = 1 + 2\tan^2\beta$, show that $\cos 2\beta = 1 + 2\cos 2\alpha$.

87. Solve the equations:

(i.) $\sin 11\theta + \sin 5\theta = \sin 8\theta$;
(ii.) $\cos 5\theta + \cos 3\theta + \cos\theta = 0$;
(iii.) $\cos^2\theta + \sin\theta - 1 = 0$;
(iv.) $\tan^2\theta + \cot^2\theta = 2$;
(v.) $(\cos\theta - \sin\theta) = \cos\alpha - \sin\alpha$;
(vi.) $5\sin\theta = 4\sin(\theta+\phi)$, $7\sin\theta = 4\sin\phi$;
(vii.) $8\cos\theta + 12\cos\phi = 13$, $12\sin(\theta+\phi) = 13\sin\theta$.

88. If $\tan\theta=\frac{b}{a}$, and $a=r\cos\theta$, show that $r=\sqrt{(a^2+b^2)}$.

89. Plot the graphs $y=\sin x$, $y=\cos x$ from 0° to 360°; and the graph $y=\tan x$ from 0° to 180°. (Use inch-squared paper, one horizontal tenth inch for 6°, and one vertical inch as unity.)

90. Show from the form of your graphs that $\cos(90°+\alpha)=-\sin\alpha$; $\sin(90°+\alpha)=\cos\alpha$.

91. $\text{Sin}\,\theta=2^n\sin\frac{\theta}{2^n}\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\ldots\ldots\cos\frac{\theta}{2^n}$. (Euler's theorem.)

92. Calculate the distance of the visible horizon from a point (i.) 20 ft., (ii.) 200 ft., (iii.) 2000 ft. above the sea-level. (Treat the earth as a circle, and calculate the lengths of tangents from the points. Take for earth's radius 3960 miles.)

93. Calculate the length of a degree at the equator.

94. If a chord of 1″ is placed in a circle of 1·5″ radius, what is the length of its arc? (Calculate the sine of half the angle at the centre.)

95. An arc of 5″ subtends an angle of 40° at the centre of a circle of radius 7·16″. What angle does an arc of 5″ subtend when the radius is 3″?

96. Calculate 30°, 45°, 150°, 225° in radians. (Use π.)

" $\frac{2\pi}{5}, \frac{7\pi}{18}, \frac{\pi}{6}, \frac{2\pi}{9}$ in degrees.

97. If θ is the circular measure of an angle, $\tan\theta>\theta>\sin\theta$.

98. If a is any chord of a circle, radius r, the length of its arc is $2r\sin^{-1}\frac{a}{2r}$. (Put $\sin\theta=\frac{a}{2r}$, then $\sin^{-1}\frac{a}{2r}=\theta$, and θ is the half angle of arc.)

99. Calculate the radius of a circle at whose centre a length of 1 cm. of arc subtends an angle of 1°.

100. If n is an integer, and α measured in radians, show that $\cos(2n\pi\pm\alpha)=\cos\alpha$; $\sin(n\pi-\alpha)=\sin\alpha$, n odd, and $\sin(n\pi+\alpha)=\sin\alpha$, n even; $\tan(n\pi+\alpha)=\tan\alpha$.

101. Give six solutions of each of the equations:

(i.) $\text{Sin}\,\theta=\frac{1}{4\cos\theta}$.

(ii.) $\text{Sin}\,\theta=3\cos\theta$.

(iii.) $4\cos^2\theta+4\cos\theta-3=0$.

PART III.

CHAPTER VII.

MODERN GEOMETRY.

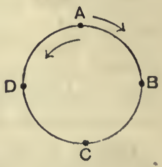
INVERSION—HARMONIC AND POLAR PROPERTIES—CROSS RATIOS—INVOLUTION—PERSPECTIVE.

Definition 1.—The **sign of a ratio** PA : PB, in which any line AB is divided by a point P, is positive or negative according as the directions PA and PB are the same or opposite—i.e. as P is external or internal to AB.

Definition 2.—Any arrangement of points in a straight line forms a **range**, and any arrangement of concurrent lines a **pencil.**

Definition 3.—A **transversal** is a straight line cutting any system of lines (curved or straight).

Definition 4.—The **cross ratio of a four-point range** ABCD **is** the ratio $\frac{AB\,.\,CD}{AD\,.\,CB}$, written (ABCD).



This may be regarded as the product $\frac{AB}{AD} \times \frac{CD}{CB}$, or as the ratio AB . CD : AD . CB.

The 1st and 3rd points A, C are **conjugates,** and the 2nd and 4th points B, D are **conjugates.**

Note. To write down the ratio (ABCD), set the points in order round a circle, and start from the first in the order of the letters for numerator, and in the reverse order for denominator.

Definition 5.—The **cross ratio of a four-way pencil** is that of a transversal range of the pencil.

Definition 6.—A **range or pencil is harmonic** when it is divided harmonically; in this case (ABCD) = – 1, and A, C are **harmonic conjugates** of B, D, and conversely.

Definition 7.—The **polar of a point to a circle or conic** is the locus of its harmonic conjugate with respect to the points in which any transversal through it cuts the circle.

The point is the **pole** of its polar.

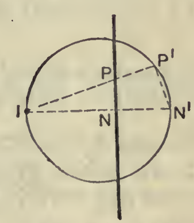
Definition 8.—Two points P, P′ are **inverse** when **PP′** passes through a fixed point **I**, the **centre of inversion,** and $IP \,.\, IP' = k$, a constant in magnitude and sign, the **constant or factor of inversion.**

If $k = a^2$, a is the **radius of inversion.** In this case k must be positive.

Two figures are inverse when the points P, P′, in which any ray through a fixed point I cuts them, are inverse.

Ex. Two circles are inverse figures with respect to a centre of similitude.

Theorem 1.—'The inverse of a straight line is a circle through the centre of inversion.'



If I is centre, k constant of invn.,
IN perp. to given line PN, and N′, P′ inverse points of N, P;
then $IN \,.\, IN' = k = IP \,.\, IP'$;
and tr. IP′N′ ||| INP,
∵ $IN' : IP = IP' : IN$; ang. I common;
∴ ang. IP′N′ = INP, a rt. ang.;
∴ locus of P′ is the circle on diam. IN′.

Note. If I is on PN, the inverse is that line; and the inverse of the centre I on a figure is the point at infinity on the tangent to the figure at I.

Theorem 2.—'The inverse of a circle is in general a circle, the centre of inversion being a centre of similitude.'

The proof is similar to that of Th. 78, Ch. IV.

Note. If the constant of inversion is the square of the tangent to a circle from the centre of inversion, the inverse of the circle is the same circle.

Ex. 1. A pencil of straight lines inverts into a system of coaxial circles.

Ex. 2. A tangent to a circle inverts into a circle touching the first circle.

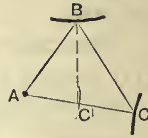
Ex. 3. When is the inverse of a straight line a straight line?

Ex. 4. When is the inverse of a circle not a circle? What is the inverse in this case?

Construction 1.—'Inscribe a triangle, given the area and one angle, on given curves.'

By inversion, we derive this construction from that for triangles of given form. (Ch. V., p. 143.)

Thus, if A is the given ang., $\triangle$ the given area, of the triangle ABC,



make $AC'.AC = k$, a constant;

then, if C moves on any curve, C′ describes its inverse, which is therefore a given curve.

Also $\dfrac{AC'}{AB} = \dfrac{AC'.AC \sin A}{AB.AC \sin A} = \dfrac{k \sin A}{2\triangle} = \text{constant};$

hence the triangle ABC′ has a given form, and can be inscribed on the curves B, C′.

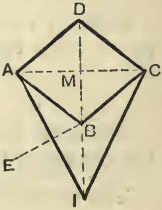
$\therefore$ the triangle ABC of given area can be inscribed on the given curves.

Ex. Inscribe a triangle, one angle 30°, area 1 sq. in., one vertex given, in an equilateral triangle of 2″ side.

Theorem 3.—'ABCD is a jointed rhombus; opposite points A, C are joined by equal rods to a fixed point I; then, if B describes any curve, D describes its inverse.' (Peaucellier.)

If AC meets BD in M,

I, B, D, each equidistant from A, C, lie on MB, the rt. bisector of AC;



and M is the mid point of DB;

$\therefore$ rect. $IB.ID = IM^2 - BM^2$

$= IC^2 - BC^2 = \text{constant};$

i.e. D is the inverse of B, constant IB . ID.

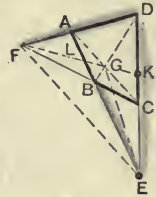
If this system is constructed in wood or metal, the inverse of any curve can be traced.

Note. If B moves on a circle, centre E, passing through I, the inverse of this circle is a straight line (Th. 1), so that in this case D describes a straight line.

We thus have a mechanical construction for a straight line, by making an additional rod EB, equal to EI, turning about a fixed point E.

(This construction is not, however, independent of the *plane*, since the circle on which B moves is a plane figure; if B is not constrained to move in a plane, it will describe any number of points on a *sphere*, centre E, and its inverse D will describe a number of points in a plane.)

Theorem 4.—'Any side of a quadrilateral is harmonically divided by the opposite side and the join of the intersection of the other two sides to that of the two diagonals.'



In the quadl. ABCD the side CD is cut in E, K by AB and FG. Then, in tr. FCD, FK, CA, DB are concurrent, and EBA is a transversal;

$$\therefore \frac{CK}{DK}\cdot\frac{DA}{FA}\cdot\frac{FB}{CB} = -1 = -\frac{CE}{DE}\cdot\frac{DA}{FA}\cdot\frac{FB}{CB};$$

$\therefore$ CK : DK = − CE : DE;

$\therefore$ (ECKD), and similarly (EBLA), is a harmonic range.

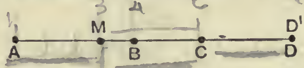
Note. FADCEB is called a **complete** quadl., with **three** diagonals AC, BD, EF. Remember that the pencils at the additional points E, F, G are harmonic.

Theorem 5.—'If M is the mid point of a pair of conjugates A, C in a harmonic range ABCD, then'—

(i.) $MB \cdot MD = MC^2 = MA^2$;

(ii.) $MB \cdot BD = AB \cdot BC$; $MD \cdot BD = AD \cdot CD$;
(B in each term); (D in each term);

(iii.) $MB : MD = AB^2 : AD^2 = BC^2 : CD^2$.



(i.) $AB \cdot CD = AD \cdot BC$, $\because AB : AD = BC : CD$;

$\therefore (MC + MB)(MD - MC) = (MC + MD)(MC - MB)$;

$\therefore 2MB \cdot MD = 2MC^2 = 2MA^2$.

(ii.) $MB \cdot BD = MB(MD - MB) = MC^2 - MB^2$
$= (MC + MB)(MC - MB) = AB \cdot BC$.

Similarly, $MD \cdot BD = MD^2 - MC^2 = AD \cdot CD$.

(iii.) $$\left(\frac{MB}{MD}\right) = \frac{MB \cdot BD}{MD \cdot BD} = \frac{AB \cdot BC}{AD \cdot CD} = \frac{AB^2}{AD^2} = \frac{BC^2}{CD^2}.$$

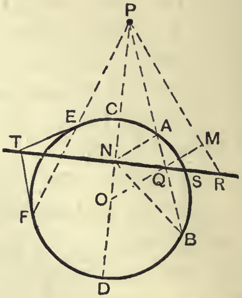
Cor.—'If $MB \cdot MD = MC^2 = MA^2$, then A, C are harmonic conjugates of B, D.'

For make D_1 harm. conj. of B to A, C;

$\therefore MB \cdot MD_1 = MC^2 = MB \cdot MD$, and D coincides with D_1.

Theorem 6.—'**The polar of a point to a circle is a straight line, and is perpendicular to the central ray of the point.**'

If PAB is a transvl. of P, and PCD a transvl. through centre O of a circle; divide PCD, PAB harmly. at N, Q, so that N, Q are on polar of P.



Then NC : CP = DN : DP = μ, say;

∴ the circle (diam. CD) is locus of points whose distances from N, P have ratio μ. (Th. 61, Ch. III.)

∴ AN : AP = μ = BN : BP;

i.e. AN : BN = AP : BP = QA : BQ;

∴ PN, QN bisect angs. N in tr. ANB, (Th. 43, Ch. III.);

∴ QN $\perp$ PN; also N is a fixed point;

∴ locus of Q—i.e. polar of P—is a straight line, perp. to OP.

Note. Calling N, the point on the central ray of P, the **foot** of the polar; then since ON . OP = OC^2,

Cor. (i.).—'**A point and the foot of its polar to a circle are inverse points to the centre, constant the square of radius.**'

Cor. (ii.).—'**The polar of a point on a circle is its tangent.**'

Theorem 7.—'**If Q is on the polar of P to a circle, P is on the polar of Q.**'

Make PM perp. to OQ; ∴ tr. OMP ||| ONQ.

∴ OM : ON = OP : OQ;

∴ OM . OQ = ON . OP = OC^2;

∴ M is foot of polar, and PM the polar, of Q.

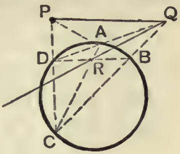
Theorem 8.—'**The polar of any point to a circle passes through the points of contact of tangents from the point; and through the intersection of the two tangents of any transversal of the point.**'

If the polar of P meets the circle at S;
polar of S—i.e. the tangt. at S—passes through P.

If TE, TF are tangents at the ends of transversal PEF,
EF through points of contact is polar of T, and traverses P;
∴ polar of P passes through T.

Definition 9.—**A self-polar triangle of a circle or conic** has each side the polar of the opposite vertex (as PQR above).

Construction 2.—'Construct the polar of a point P to a circle or conic.'*



Draw two transversals PAB, PDC, join AC, DB and AD, BC to meet in R, Q;
∴ QR divides PAB, PCD harmly. in the quadl. ABCD;
∴ QR is the polar of P.

Note. PQR is a self-polar triangle, and if P, Q, two vertices of a self-polar triangle, are joined to a point A on the curve, and produced to D, B, and PD produced to C; then CB meets DA on polar of P—i.e. in Q; and DB passes through the 3rd vertex R. Hence:

Cor.—'Transversals through a point on a circle or conic and through two vertices of a self-polar triangle determine a chord through the third vertex.'

Ex. Construct the pole of a line. (Draw polars of two points.)

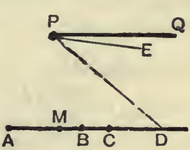
Point at Infinity.

If P, P′ are inverse points in a line, centre I, constant k, to every P corresponds one P′; and if IP is small enough, IP′ is greater than a given length; hence, if P coincides with I, IP′ is greater than every length—i.e. infinite; and we say,

'A straight line has one point at infinity.'

Examples. In Th. 5, if B coincides with M, its harm. conj. D is at infinity, and $DA : DC = BA : CB = 1$; hence,

'The point at infinity on AB bisects AB externally.'



If ABCD is a harm. range, M the mid point of AC, PQ parl. to AB; then if B coincides with M, MD is infinite, and PD coincides with the parl. PQ since any other line PE cuts AB. Hence,

'Two parallels meet in one point at infinity.'

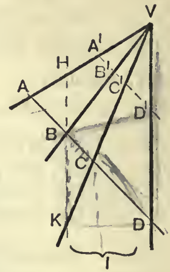
If HK, Th. 9, meets VD at inf. in I; then $(HBKI) = \dfrac{HB.KI}{HI.KB} = \dfrac{HB}{KB} = (ABCD)$.

Hence Th. 9 is true when the transversal ‖ one ray.

Ex. 'The limit of a circle through a fixed point A, whose centre O moves to infinity along a fixed line AO, is a perpendicular line.'

* In Ch. VIII. we show that conics have the polar properties of a circle.

Theorem 9.—'The cross ratio of a fourway pencil VA, VB, VC, VD, in a given order, is constant.'



If ABCD is a transversal, draw HBK parl. to VD;

$$\therefore \frac{AB}{AD} = \frac{HB}{VD}, \quad \frac{CD}{CB} = \frac{VD}{KB};^{*}$$

$$\therefore (ABCD) = \frac{AB \cdot CD}{AD \cdot CB} = \frac{HB}{KB}.$$

Similarly, any transvl. through B has cross ratio HB : KB; and any other transvl. || one of these, and has the same cross ratio;

∴ the cross ratio of all transversals is the same.

Theorem 10.—(i.) 'The value μ of a cross ratio is unaltered if the order of points is reversed, or if the points of each pair of conjugates are interchanged.'

(ii.) 'μ is changed into $\frac{1}{\mu}$ if the points of one pair of conjugates are interchanged.'

For (i.) if $(ABCD) = \mu$, reverse the order;

then $(DCBA) = \dfrac{DC \cdot BA}{DA \cdot BC} = \dfrac{AB \cdot CD}{AD \cdot CB} = \mu.$

Interchange the points A, C and B, D of conjugates;

then $(CDAB) = \dfrac{CD \cdot AB}{CB \cdot AD} = \mu = (BADC)$, reversing.

Note. The 24 possible orders of points A, B, C, D give only 6 different cross ratios, since they are equal in fours.

(ii.) Interchange A, C or B, D only;

$\therefore (CBAD) = \dfrac{CB \cdot AD}{CD \cdot AB} = \dfrac{1}{\mu} =$ similarly (ADCB).

Note. The six values due to different orders of the points can be derived from HBK above. Make $KB = 1$, $HB = \mu$; then we have

μ	{i.e. (ABCD)}, and $1/\mu$,	dividing	HK at B,
$1-\mu$,	{(ACBD)}, and $1/1-\mu$,	"	HB at K,
$\mu/\mu-1$,	{(ABDC)}, and $(\mu-1)/\mu$,	"	BK at H.

These are easily obtained by using the point at infinity on the parl. transvl.

* Care must be taken to make all equalities true in sign.

Theorem 11.—'A range or pencil is harmonic when, and only when, its cross ratio is unaltered by interchange of one pair of conjugates.' *

If $(\mathsf{ABCD}) = \mu$, and is unaltered by interchanging A, C;

$\mu = (\mathsf{CBAD}) = \frac{1}{\mu}$, and $\mu^2 = 1$.

$\therefore \mu = \pm 1$.

But no cross ratio can have value $+1$, since then $\mathsf{AB} : \mathsf{AD} = \mathsf{CB} : \mathsf{CD}$, and either A, C or B, D coincide.

$\therefore \mu = -1$, and the range is harmonic.

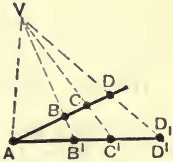
And if the range is harmonic, $\mu = -1$;

$\therefore (\mathsf{CBAD}) = \frac{1}{\mu} = -1 = (\mathsf{ABCD})$;

i.e. a pair of conjugates A, C can be interchanged without altering the value of the cross ratio.

Definition 10.—Two ranges or pencils A, B, C..., A′, B′, C′... are **homographic** which have the cross ratio of any four elements A, B, C, D of one equal to that of the corresponding elements A′, B′, C′, D′ of the other.

Theorem 12.—'Two homographic ranges which have one pair of corresponding points coincident, have the joins of all other pairs of corresponding points concurrent.'



If homographic ranges ABCD, AB′C′D have A common, and $(\mathsf{ABCD}) = (\mathsf{AB'C'D'})$;

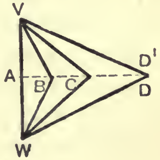
join BB′, CC′ to V, and VD to meet AB′ in $\mathsf{D_1}$.

$\therefore (\mathsf{AB'C'D_1}) = (\mathsf{ABCD}) = (\mathsf{AB'C'D'})$;

$\therefore \mathsf{C'D_1} : \mathsf{AD_1} = \mathsf{C'D'} : \mathsf{AD'}$;

$\therefore \mathsf{D_1}$ coincides with D′, and D′D traverses V.

Theorem 13.—'Two homographic pencils which have one pair of corresponding rays coincident, have the intersections of all other pairs of corresponding rays collinear.'



If homg. pencils V(ABCD), W(ABCD′) have ray VAW common, and $\mathsf{V(ABCD)} = \mathsf{W(ABCD')}$;

draw BCA cutting VD, WD′ in D, D′.

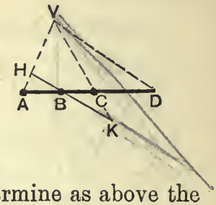
$\therefore (\mathsf{ABCD'}) = (\mathsf{ABCD})$, and D′ coincides with D;

i.e. the pair VD, WD′ intersect on BC.

* This theorem gives a very simple proof of Th. 4, above.

Construction 3.—'Given three elements in a range or pencil of given cross ratio μ, determine the fourth.'

If A, B, C are three points in a range, draw HBK, making HB : KB eql. to μ; draw AH, KC to V, and VD parl. to HK.



Then if I is the point at inf. on HK,

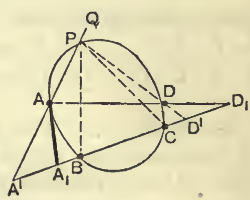
$$V(ABCD) = V(HBKI) = \frac{HB \,.\, KI}{HI \,.\, KB} = \frac{HB}{KB}$$
$$= \mu.$$

If VA, VB, VC are three rays in a pencil, determine as above the fourth point D on any transversal; then VD is the fourth ray.

Ex. Show that there is only one solution.

Construction 4.—'Construct the locus of a point whose pencil at four fixed points, no three collinear, has a given cross ratio μ.'

If A, B, C, D are the points, draw any ray AP to A′ on BC; make (A′BCD′) eql. to μ, and join D′D to P on AA′.



$\therefore$ P(ABCD) = (A′BCD′) = μ.

$\therefore$ P is a point on the curve.

And any third point Q on AP has a different cross ratio Q(ABCD); hence one only point (other than A) can be found on every ray through A; and the curve can be plotted by finding a number of points.*

We may call this curve a **cross ratio curve** of the points.

Theorem 14.—'A cross ratio curve of four fixed points, no three collinear, passes through each point.'

If A, B, C, D (last fig.) are the fixed points, P a point on the curve, draw AD to D_1, make (A_1BCD_1) eql. to $\mu = P(ABCD)$.

Then, if AP turns about A to coincidence with A_1A,

A′, D′ coincide with A_1, D_1,

D′P coincides with D_1A, and P with A.

Hence A is a point on the curve.

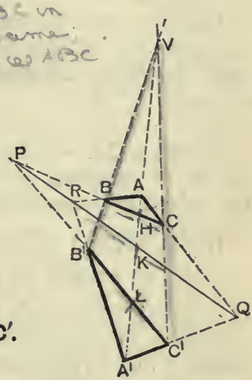
Any transversal through A cuts the curve in two only points, and AA_1 through two coincident points at A is the tangent at A.

We show in the next chapter that this curve is a conic.

* Pascal's theorem gives a simple construction for the point P.

Theorem 15.—'If the joins AA', BB', CC' of vertices of two triangles are concurrent, the intersections P, Q, R of corresponding sides BC, $B'C'$, &c. are collinear; and conversely.' (Desargues.)

If BB', CC' meet in V, and AA' cuts BC, PQ, $B'C'$ in H, K, L; then



(i.) If AA' passes through V, pencil $V(PBAC)$ is cut by BC, $B'C'$,

$\therefore\ (PBHC) = (PB'LC')$;

$\therefore\ A(PBHC) = A'(PB'LC')$.

These pencils have HL common,

$\therefore$ the intersections of corresp. rays are collinear;

i.e. P, R, Q are collinear.

(ii.) If R is on PQ,

$A(PRKQ) = A'(PRKQ)$;

$\therefore\ (PBHC) = (PB'LC')$ on transvls. BC, $B'C'$.

These ranges have P common,

$\therefore$ joins of corresp. points are concurrent;

i.e. BB', AA', CC' are concurrent.

This is the fundamental theorem in perspective.

Definition 11.—Two figures in a plane are **in perspective** which have the joins of corresponding points concurrent (BB', AA', CC') in the **centre of perspective** (V), and the intersections of corresponding sides collinear on the **axis of perspective.** (BC, AC, AB)

Construction 5.—'Construct the perspective of a given figure $ABC\ldots$, to centre V, axis PQ.'

Choose A' on VA to correspond to A, produce AB to meet the axis in R,

join RA' to meet VB in B';

$\therefore$ B' is the point corresp. to B,

and $A'B'$ " side " AB.

Similarly any number of points, sides, or chords may be constructed.

Ex. Show that the triangles RBB', CQC' in the above figure are in perspective. Hence derive (ii.) of Th. 15 from (i.).

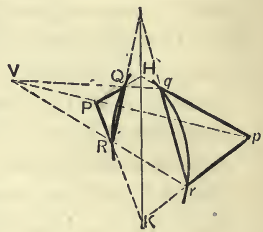
The process of deriving a figure as the perspective of another in a plane is called **plane projection**, or simply projection.

Orthogonal or right projection is a particular case of this, in which all rays through the vertex are parallel (the vertex being at infinity), and perpendicular to the axis of perspective.

'Any polygon projects into a polygon, a curve into a curve, a chord into a chord, a transversal into a transversal, a tangent (limiting transversal) into a tangent, and a pencil into a pencil.'

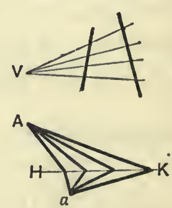
Theorem 16.—'The intersections of corresponding pairs of chords or tangents of two figures in perspective are corresponding points in perspective.'

If PQ, PR and *pq*, *pr* are corresp. pairs of chds. or tangts. in persp., vertex V, axis HK, meeting at P, *p*;
then QR, *qr* are also corresp. chds., and intersect on the axis HK.
∴ trs. PQR, *pqr* are in persp.,
and P*p*, conc. with Q*q*, R*r*, traverses V;
i.e. P, *p* are corresp. points in perspective.



Theorem 17.—'Corresponding ranges or pencils in perspective are homographic.'

For two ranges in persp. are formed on transversals of a common pencil;
and two pencils in persp. have a common axial transversal.



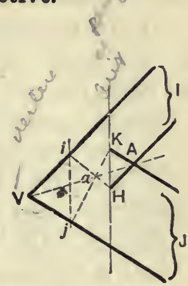
Ex. 'The polar of a point to any perspective of a circle is a straight line.'

Note. We show in the next chapter that every conic is the perspective of a circle, and conversely; we are thus able to deduce many properties of the conic—e.g. its polar and cross-ratio properties—from those of the circle.

The properties of the line at infinity are most important in projection. The line of one figure which projects to infinity in its perspective is called the **vanishing line** of the first figure, because it does not appear in the second.

Theorem 18.—'Points at infinity in a figure project into a straight line parallel to the axis in its perspective.'

If V, HK are vertex and axis of persp., a the persp. of A, and K, A, a fixed points, and HA any line through A in fig. A; then parls. Vi, HA and Vj, KA meet in points I, J at inf. in fig. A; hence i, j on Ha, Ka are the projns. of points at inf. I, J in fig. A.



Also, j is fixed, ∵ KA, Ka, Vj are fixed; and $ia : a\text{H} = \text{V}a : a\text{A} = ja : a\text{K}$;

∴ $ij \parallel$ HK; i.e. locus of i is a fixed parl. to HK.

Cor.—'Points at infinity in a figure form a straight line.' * (The straight line at infinity.)

Note. ij is the vanishing line of fig. a.

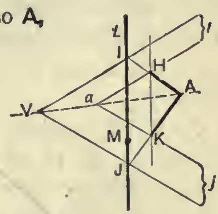
Ex. 'A parallelogram projects into a complete quadrilateral.'

Construction 6.—'Project a figure so that a given straight line may be projected to infinity, and any two angles into given angles.'

If IJ in fig. A is to project to inf.; choose any vertex V, any point a corresp. to A, and make aH, aK parl. to VI, VJ;

∴ $\text{IH} : \text{HA} = \text{V}a : a\text{A} = \text{JK} : \text{KA}$;

∴ HK, axis of persp., $\parallel$ IJ.



Also, i in fig. a is on aH, VI; i.e. i is at inf. and is projn. of I; simly., j is at inf. and is projn. of J;

∴ IJ in fig. A projects to inf. in fig. a.

(ii.) Ang. HaK = IVJ. Hence, by choosing V on an arc IVJ of given ang., the ang. HaK = given ang. Thus the ang. IAJ can be projected into any given angle.

Also, by making two arcs IVJ, LVM, of given angles, to meet in V, we can project any two angles IAJ, LAM at A into given angles at a.

Note. IJ is the vanishing line of fig. A.

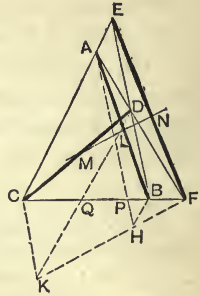
Ex. Project the angles formed by a given angle and its bisectors into two right angles. Also project any quadrilateral into a rectangle.

* They also form circles and other curves.

Theorem 19.—'The mid points of diagonals of a complete quadrilateral are collinear.'

If L, M, N are mid points of diags. AB, CD, EF of a quadl.; make APH, CK parl. to BE, and DQK, BH parl. to CE.

Then M, L are diag. points of parms. EK, EH.



Also $\frac{FB}{BP} = \frac{FD}{DA} = \frac{FQ}{QC}$;

$\therefore \frac{FB}{FQ} = \frac{BP}{QC} = \frac{BH}{QK}$ (simr. trs.);

$\therefore$ KHF is a str. line;

also M, L, N bisect EH, EK, EF;

$\therefore$ L, M, N are collinear.

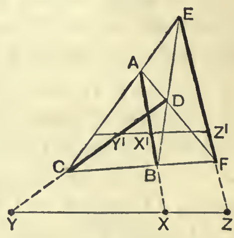
The following generalisation by projection is a good illustration of the process, and also of the importance of the line at infinity.

Theorem 20.—'If three points on the diagonals of a complete quadrilateral are collinear, their harmonic conjugates are also collinear.'

If X, Y, Z are collinear points on diags. AB, CD, EF of a quadl., and X′, Y′, Z′ their harm. conjs.; project the figure so that XYZ projects to infinity;

$\therefore$ X′, Y′, Z′ project into mid points of diags. L, M, N of the new fig. (see above), which are collinear;

$\therefore$ X′, Y′, Z′ are collinear.



PERSPECTIVE OF A CIRCLE.

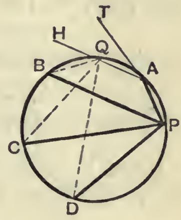
In any conic or circle, every central chord is bisected at the centre; hence the harm. conj. of the centre bisects each chord externally and is at infinity; the polar of the centre is therefore entirely at infinity, i.e.—

'The centre of a conic or circle is the pole of the line at infinity.'

We may define the **centre** of a **perspective of a circle** as the pole of the line at infinity, and **diameters** as central chords.

Theorem 21.—'The cross ratio of the pencil from any point on a circle or conic to four fixed points on it is constant.'

If A, B, C, D are the fixed points, and if P is on arc AD, Q on another arc AB, and AQ is produced to H;
then ang. HQB = suppt. of AQB = APB;
ang. BQC = BPC; CQD = CPD;
∴ pencil Q(ABCD) ≡ P(ABCD).



Thus, wherever P is in AD, and simly. in any other arc,
P(ABCD) = Q(ABCD) = constant.

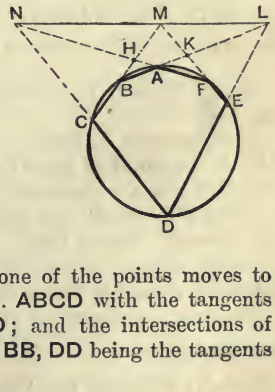
Also, if Q moves up to coincidence with A, AQ becomes the tangent AT;
∴ A(ABCD) = A(TBCD) = P(ABCD).

Thus the cross ratio has the same value when the fifth point coincides with one of the four.

Note. By projection this theorem is true for any perspective of a circle.

Theorem 22.—Pascal's theorem: 'The intersections of opposite sides of an inscribed hexagon of a circle or conic are collinear.'

If the opp. sides of ABCDEF in circ. meet, AB, DE in L;
BC, EF in M; CD, FA in N;
and BC, EF meet AF, AB in H, K;
then C(ABDF) = E(ABDF);
∴ on transversals AN, AL,
(AHNF) = (ABLK), with A common;
∴ HB, NL, FK are concurrent;
i.e. L, M, N are collinear.

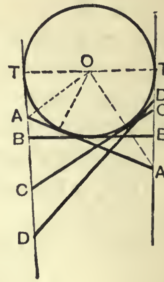


Note. This theorem remains true when one of the points moves to coincidence with the next. Thus the quadl. ABCD with the tangents BT, DT form a Pascal hexagon ABBCDD; and the intersections of BB, DD; AB, CD; BC, AD, are concurrent, BB, DD being the tangents at B, D, and T their point of intersection.

The polar properties of a circle or conic can be thus derived. (See last page of chapter.)

Theorem 23.—'The cross ratio of the range formed on any tangent of a circle or conic by four fixed tangents is constant.'

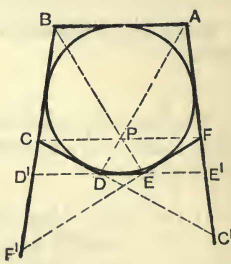
If AA′, BB′, CC′, DD′ are fixed tangents of a circle, centre O, TA, T′A′ any two tangents;
ang. AOA′ = $\frac{1}{2}$TOT′ (Ex. XXVI. 10),
= BOB′ (similarly);
∴ ang. AOB = A′OB′; BOC = B′OC′, &c.;
∴ pencil O(ABCD) ≡ O(A′B′C′D′);
∴ (ABCD) = (A′B′C′D′) = constant.



Note. By projection this theorem is true for any perspective of a circle.

Theorem 24.—Brianchon's theorem: 'The joins of opposite vertices of a circumscribed hexagon of a circle or conic are concurrent.'

If ABCDEF is a circumhexagon of a circle, and AD, BE meet in P;
the ranges in which tangts. AF, BC meet the other four are homographic;
∴ E(BCD′F′) = D(AC′E′F),
and these pencils have ray D′E′ common;
∴ intersections of corresponding rays are collinear;
i.e. P, C, F are collinear;
i.e. AD, BE, CF are concurrent.



Note. By the aid of pole and polar this can be derived from Pascal's theorem.

If tangents at A, B, C..., in Th. 22 above, form a hexagon UVWXYZ, UX is the polar of L; VY of M; WZ of N; these all traverse the pole of LMN, and are therefore concurrent.

Also, Thh. 22, 24 are true for any orders of the letters. There are 60 different Pascal lines, obtained by arranging the vertices in all possible orders. An interesting account of these lines is given in Salmon's *Conics*, p. 260 (fifth edition).

In order to apply the theorems, write down the points or sides in suitable order—e.g. $\overbrace{A\smile X\frown P\smile L\smile R\frown Z}$;* opposite sides or vertices are AX, LR; XP, RZ; and PL, ZA.

* Or round a circle.

Definition 12.—An **involution range or pencil** is a system of pairs AA', BB', CC'... of points or rays such that any four are homographic with their conjugates.

AA', BB', CC' are **pairs** or **conjugates** of the involution.

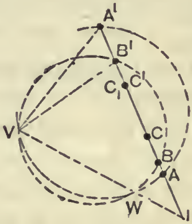
The **centre of an involution range** is the conjugate of the point at infinity.

Cor.—'Any transversal of a pencil in involution is cut in involution.'

An involution is **separate** when the points or rays of one pair are one internal and one external to those of any other pair; and **continuous** when the points or rays of one pair are both internal or both external to those of another pair.

Theorem 25.—(i.) 'Inverse points in a line to a given centre form an involution range.'

(ii.) 'Pairs of an involution range are inverse points to the centre of the range.'



(i.) If AA', BB', CC' are inverse points to centre I, const. k,
draw a circle $AA'V$, and through V the circle VBB' cutting VI in W;
$\therefore\ IW \,.\, IV = IB \,.\, IB' = IA \,.\, IA'$;
hence W is also on circle VAA';
and similarly on circ. VCC', &c.;
$\therefore$ ang. $A'VB' = VB'I - VA'B'$
$= IWB - IWA = AWB$.

Similarly, $B'VC' = BWC$, &c.;
$\therefore$ pencil $V(A'B'C'\ldots) \equiv W(ABC\ldots)$,
$\therefore\ (A'B'C'D') = (ABCD)$ &c., and $(AA', BB'\ldots)$ is in involn.

(ii.) If AA', BB', CC',... are pairs in involution,
draw circles $AA'V$, $BB'V$ to meet in W, draw VW to I;
$\therefore\ IA \,.\, IA' = IV \,.\, IW = IB \,.\, IB'$.

Find C_1 so that $IC \,.\, IC_1 = IA \,.\, IA' = IB \,.\, IB'$;
$\therefore\ AA'$, BB', CC_1 are pairs in involution, by (i.);
$\therefore\ (AA'BC_1) = (A'AB'C) = (AA'BC')$, by Def. 12;
$\therefore\ C'$ coincides with C_1; and $IC \,.\, IC' = IA \,.\, IA'$, &c.;
$\therefore\ I$, conj. of the point at inf. on AA', is centre of the involution.

Cor.—'Two pairs AA', BB' determine an involution.'

Ex. Prove when A, A' are one internal and one external to BB'.

Definition 13.—The **double** $\begin{matrix}\textbf{points}\\ \textbf{rays}\end{matrix}$ of a $\begin{matrix}\text{range}\\ \text{pencil}\end{matrix}$ in involution are self-conjugate $\begin{matrix}\text{points}\\ \text{rays}\end{matrix}$; i.e. each is its own conjugate.

Definition 14.—**Real points** in a plane are points whose distances from some given point are measured by a real number μ; and **real lines** are lines having real points continuously to infinity in either direction.

Real lines and points can be constructed by ruler, compass, &c.

Imaginary points in a plane are points whose distances from some given point are measured by $\mu\sqrt{-1}$, where μ is some real number; and **imaginary lines** in a plane through a given point are lines all of whose points are imaginary except the given point.

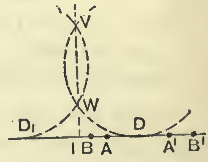
Imaginary lines and points can be *imagined*, but not constructed by the ordinary constructions for real points.

Ex. $\sqrt{-1}$ is represented by a turn through a right angle. Justify this.

Theorem 26.—'A $\begin{matrix}\textbf{range}\\ \textbf{pencil}\end{matrix}$ in involution has one pair of double $\begin{matrix}\textbf{points}\\ \textbf{rays}\end{matrix}$; these are harmonic conjugates of any pair, and are real or imaginary according as the involution is continuous or separate.'

If I is centre of involn. range or transvl.:

(i.) If the involn. is continuous, circles AA′V, BB′V meet in points V, W on the **same** side of AB (1st fig.); and I is external to AA′ and to BB′.



Draw circs. VW touching AB at D, D_1;

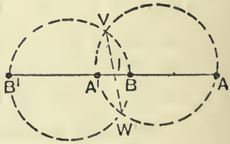
$\therefore\ ID^2 = ID_1{}^2 = IA \,.\, IA'$, which is positive,

$= +\mu^2$, μ a real measure.

$\therefore\ ID = +\mu = -ID_1$.

Thus D, D_1 are real points.

(ii.) If the involn. is separate (2nd fig.), circs. AA′V, BB′V meet on **opp.** sides of AB; and I is internal to AA′ and to BB′.



Hence if δ, δ_1 are double points,

$I\delta^2 = I\delta_1{}^2 = IA \,.\, IA'$, which is negative,

$= -\mu^2$, where μ is a real measure;

$\therefore\ I\delta = \mu\sqrt{-1} = -I\delta_1$; and δ, δ_1 are imaginary.

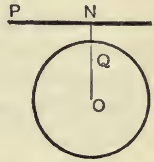
Since $IA \,.\, IA' = ID^2$ or $I\delta^2$; D, D_1 or δ, δ_1 are harm. conjs. of A, A′.

Imaginary points enable us to make our theorems continuous. Thus we say:

'A straight line meets a circle or conic in two real, two coincident, or two imaginary points;'

so that a straight line always meets a circle or conic in two points.

Construction 7.—'Determine the imaginary points in which a line outside a given circle meets it.'



If PN cuts a circ., centre O, in δ, δ_1, if ON $\perp$ PN, and r is measure of radius;

$$r^2 = \mathrm{O}\delta^2 = \mathrm{ON}^2 + \mathrm{N}\delta^2;$$

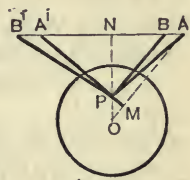
$\therefore\ \mathrm{N}\delta^2 = r^2 - \mathrm{ON}^2 = -\mu^2$, say, μ real;

thus $\mathrm{N}\delta = \sqrt{-1}\,\sqrt{\mathrm{ON}^2 - r^2} = -\mathrm{N}\delta_1$.

Ex. If Q is pole of PN, then $\mathrm{Q}\delta$, $\mathrm{Q}\delta_1$ are tangents from Q. (Calc. $\mathrm{Q}\delta^2$.)

Theorem 27.—'The pencil of sides of self-polar triangles of a circle at a common vertex is in involution.'

'A line in the plane of a circle is cut in involution by the sides of self-polar triangles of its pole.'



If P is the vertex of self-polar trs. PAA$'$, PBB$'$, &c., of a circle, centre O, and N the foot of its polar AA$'$,

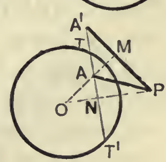
draw A$'$PM perp. to OA, (polar of A);

then $\mathrm{NA} : \mathrm{ON} = \mathrm{MP} : \mathrm{MO} = \mathrm{NP} : \mathrm{NA}'$;

$\therefore\ \mathrm{NA}\,.\,\mathrm{NA}' = \mathrm{ON}\,.\,\mathrm{NP}$ (in magn. and sign)

$= \text{simly. } \mathrm{NB}\,.\,\mathrm{NB}'$, &c.;

i.e. P(AA$'$, BB$'$...) and (AA$'$, BB$'$...) are in involn.



Note. If P is a point outside the circle, from which tangents PT, PT$'$ can be drawn, T, T$'$ are the points in which AA$'$, polar of P, meets the circle; also, if A moves to coincidence with T, A$'$ coincides with T, and PT and T each is its own conjugate.

If P is inside circle and δ, δ_1 double points of (AA$'$, BB$'$...),

$$\mathrm{N}\delta^2 = \mathrm{ON}\,.\,\mathrm{NP} = -\mathrm{ON}(\mathrm{ON} - \mathrm{OP}) = r^2 - \mathrm{ON}^2. \quad \text{Hence,}$$

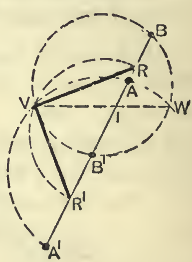
Cor.—'The double $\genfrac{}{}{0pt}{}{\text{points}}{\text{rays}}$ of the self-polar involution of a $\genfrac{}{}{0pt}{}{\text{line}}{\text{point}}$ to a circle are its $\genfrac{}{}{0pt}{}{\text{points on}}{\text{tangents to}}$ the circle.'

These properties of self-polar triangles of a circle remain true in projection; that is, they are true of any perspective of a circle, except that the centre N of the involution on AA′, though on the central ray through P, the pole of AA′, is not generally the foot of the perpendicular from the centre or the point P on the line AA′.

On account of their frequent occurrence in conics, we shall use the following abbreviations:

s.p. tr. for self-polar triangle;
s.p. involn. of a point for the involution of sides of self-polar triangles with this point as vertex;
s.p. involn. of a line for the range on it of the s.p. involn. of its pole.

Theorem 28.—'Every pencil in involution has one right-angled pair, and is entirely right-angled when two pairs are right-angled.'



If AA′ is a transvl. of the pencil in involn. V(AA′, BB′...),
draw circs. VAA′, VBB′ meeting at W;
determine I, centre of involn. (AA′, BB′...).

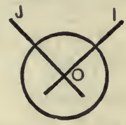
Draw circ. VWR, centre on AA′, cutting AA′ in R, R′;
∴ RVR′ is a rt. ang., ∵ RR′ is a diam.,
and IR . IR′ = IV . IW = IA . IA′, &c.
∴ VR, VR′ is a rt.-angled pair of V(AA′, BB′...).

(ii.) If a second pair VA, VA′ is rt.-angled,
AA′ is diam. of circ. VAA′;
∴ AB, line of centres of VAA′, VRR′, is rt. bisr. of VW;
∴ centre of circ. VBB′ is in AB, and BB′ its diam.;
∴ VB, VB′, and simly. any other pair, is right-angled.

Note. As any two angles at a point can be projected into right angles, any involution pencil can be projected into a rt. pencil. But the projection is real when only the circles which determine the vertex of projection (Constr. 6, ii.) meet in real points—i.e. when the involution is separate and not continuous.

Ex. Show that every right pencil is in involution.
Can the double rays of a right pencil be represented?

Theorem 29.—'The self-polar involution of the centre of a circle is right-angled.'



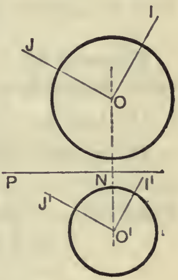
If I is the point at inf. on a central ray OI of a circle, then OJ, the polar of I, traverses O, and $\perp$ OI.

Simly., if J is at infinity on OJ,

OI is the polar of J, and IOJ a s.p. tr.;

$\therefore$ each pair of the s.p. involn. of O is rt.-angled.

Theorem 30.—'Two circles meet in four points; two at infinity, imaginary, and two other real or imaginary points.'



If O, O′ are centres of two circles, and IOJ, I′O′J′ right angles of parl. sides; then OI, O′I′, and simly. OJ, O′J′, meet at inf.;

$\therefore$ s.p. involns. of line at inf. of the circs. O and O′ are congruent;

$\therefore$ they have the same double points, which are common points of the two circles; also these common points are imaginary, because the involn. is separate.

These are the **circular points at infinity.**

The other two common points of the circles are the points, real or imaginary, in which the radical axis of the circles meets them.

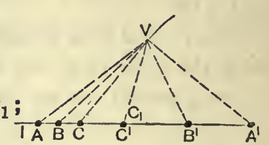
Note. As the line joining O or O′ to a circular point at infinity is a double ray of a right-angled pencil, it is its own perpendicular; hence a remarkable property of the circular points at infinity:

'The line joining any point to a circular point at infinity is perpendicular to itself.'

The construction of rays or points in involution reduces to that for a radical axis of two circles.

Thus, in Th. 25, the pairs AA′, BB′, &c. are the intersections of the transversal AA′ with the coaxial circles AVW, BVW, &c., whose radical axis is VW; and the radical axis VW is itself the limiting circle of this system, whose centre is at infinity on the right bisector of VW.

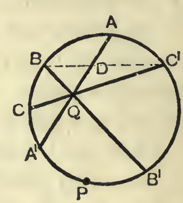
Theorem 31.—'A range / pencil of three pairs is in involution if the cross ratio of any four points / rays is equal to that of their conjugates.'



If A, A'; B, B'; C, C', are three pairs of points of the range or on a transvl. of the pencil;
and if, for example, $(AA'BC) = (A'AB'C')$,
find I, C_1 so that $IA \,.\, IA' = IB \,.\, IB' = IC \,.\, IC_1$;
$\therefore$ (AA', BB', CC_1) is in involn.,
and $(A'AB'C_1) = (AA'BC) = (A'AB'C')$;
$\therefore$ C' coincides with C_1, and (AA', BB', CC') is in involn.

Note. This is the most useful form of test for an involution, derived from its homographic property.

Theorem 32.—'A system of concurrent chords of a circle or conic subtends a pencil in involution at any point on the circle.'

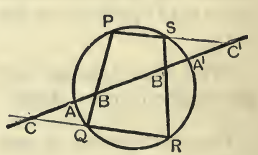


If chds. $AA', BB', CC' \ldots$ of a circle meet at Q, and BC' meets AA' in D;
then, if P is a point on the curve,

$$\begin{aligned} P(AA'BC) &= C'(AA'BC) = (AA'DQ) \\ &= B(AA'DQ) = B(AA'C'B') \\ &= P(B'C'A'A), \text{ reversing,} \\ &= P(A'AB'C'), \text{ interch. conjs.} \quad \text{(Th. 12.)} \end{aligned}$$

$\therefore$ $P(AA', BB', CC' \ldots)$ is in involution.

Theorem 33.—'A transversal is cut in involution by a circle or conic and the opposite sides of an inscribed quadrilateral.'



If transvl. AB cuts circle and sides of quadl. $PQRS$ in A, A', B, B', C, C';

$$\begin{aligned} (AA'BC') &= P(AA'BC') = P(AA'QS) \\ &= R(AA'QS) = (AA'CB') \\ &= (A'AB'C). \quad \text{(Th. 12.)} \end{aligned}$$

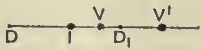
Note. These theorems, 32 and 33, can be proved in exactly the same way for a conic.

Ex. Show in the same way that the diagonals PR, SQ determine a pair in the involution. What property of a quadrilateral can you deduce? Of a system of conics through four points? What will the double points represent in this system?

Construction 8.—'Construct the rectangle of parts from any point of a transversal to a circle or conic.'

(ii.) **'Construct the rectangle of the parts from the double points of an involution range to any point in the range.'**

If V is a point on a transvl. of a circle or an involn. range, and if I is the centre of the given involn. or of the s.p. involn. of the transvl., determine V′, conj. of V.

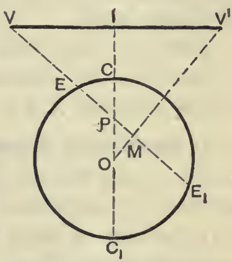


(i.) The involn. has real double points D, D_1, intersections of transvl. and curve;

∴ D, D_1 are harm. conjs. of V, V′. (Th. 29.)

∴ VD . VD′ = VI . VV′. (Th. 5, ii.)

(Note that V is first letter in each term.)



(ii.) The involn. has imaginary double points δ, δ_1, intersections of transvl. and curve.*

Construct two points P, O (P between I, O) on the perp. IO, such that OI . IP = IV . IV′ (I in each term); find C, C_1 such that $OC^2 = OP \,.\, OI = OC_1{}^2$,

draw circ., centre O, rad. OC;

∴ VV′ is polar of P to this circle;

also IV . IV′ = OI . IP;

∴ V, V′ are a pair in the s.p. involn. of VV′ to circle (Th. 27.), and VPV′ is a s.p. triangle.

∴ δ, δ_1 are the intersections of VV′ and circle.

Hence, if VP cuts circle in E, E_1,

$V\delta \,.\, V\delta_1 = VE \,.\, VE_1$ (rect. property of circle)

$= VP \,.\, VM$, by (i.) (E, E_1 double points on transvl. VP),

$= VI \,.\, VV'$, in cyclic quadl. IPMV′.

Hence the following simple principle:

Theorem 34.—'The rectangle of the parts from a point of a transversal to a conic or circle is equal to the rectangle of the parts from the point to its own conjugate and the centre of the self-polar involution of the transversal.'

This gives a simple proof of Appolonius' theorem, Ch. VIII., Thh. 11, 31.

* We may prove (ii.) without circle, thus: since I bisects $\delta\delta_1$,

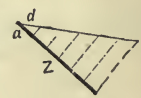
$$V\delta \,.\, V\delta_1 = VI^2 - I\delta^2 = VI^2 - IV \,.\, IV' = VI^2 + VI \,.\, IV' = VI \,.\, VV'.$$

Our treatment of similar figures in Chh. III., V. rested on certain theorems (Unit, Alternando, Summation) which we derived in Ch. V. from the properties of number. But by modifying the fundamental axiom* of magnitude, we can derive all properties of figures by **pure**—i.e. self-contained—**geometry**, without the aid of number, as follows.

Archimedes' Principle.—'Any magnitude, however small, can be so repeated that the whole exceeds any given magnitude of the same kind.'

Hence, if the difference d of magnitudes X, Y cannot be so repeated as to exceed a given magnitude Z, d must be zero magnitude, and X = Y.

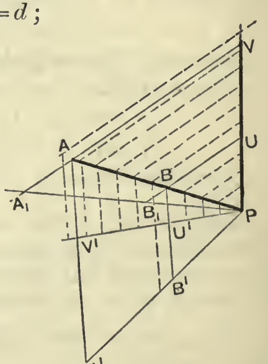
I. Repetition Theorem.—If the adjoining figure represents a mode of repetition of a length d along a line, parallels at successive points determine a like mode of repetition of some length a whose whole repeats into a given length Z. Hence:



'All modes of repetition of any length d are included in those of all lengths a which repeat into a given length Z.'

II. Absolute Theorem.—'If a constant length PU, called the absolute, is drawn from a fixed point P in a fixed line AB, and AV parallel to BU meets PU in V, then PV is constant, and is called the relative of PA to the absolute of PB.'

If PU′ = PU, AV′ ‖ BU′, and PV ~ PV′ = d;
and if a is any length repeating into PU, or PU′;
then the equidistant parls. to BU, BU′, dividing PU, PU′ into parts a, are concurrent, two and two, on PB, and have V, V′ between corresp. parls. of the two systems.



∴ d = PV ~ PV′ < a;
repeat d so that a repeats into PU;
∴ whole of d < PU;
and this is true for all values of a;
i.e. for all modes of repetition of d;
∴ d is zero, and PV = PV′ = const.

* Gr. ἀξίωμα or κοινὴ ἔννοια (common notion).

If $PA'B'$ is any second transvl. of parls. AA', BB' ($PA \not> PA'$), draw PA_1B_1 to AV, BU, so that $PB_1 = PB'$ (last fig.);
$\therefore$ $PA_1 = PA'$ (as $PV = PV'$ above);
$\therefore$ relative of PA' to abs. of PB' = rel. of PA_1 to abs. of $PB_1 = PV$.

Cor.—'The relative of one part PA of a transversal from a fixed point to two fixed parallels, to the absolute of the other part PB, is constant.'

Ex. If $PB_1 = -PB$, prove $PV_1 = -PV$.

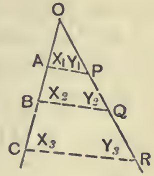
Geometrical definition of ratio.—The **ratio of a line X to a line Y** is the relative of X to the absolute of Y.
Take X as PA, constr. $PB = Y$, PU the abs.; then $PV = X : Y$.

Rule of Signs.—PU is positive; hence we see at once, drawing PB the opp. way to PA, that $X : Y = -(X : -Y) = -(-X : Y)$.

III. Unit Theorem.—(i.) 'If $X : Y = Z : Y$, then $X = Z$.'
If X alters to Z, Y const., PB, PU remain fixed, and A moves;
$\therefore$ the ratio PV alters and $Z : Y \neq X : Y$;
hence X does not alter, and Z must be equal to X.
(ii.) 'If $X : Y = X : Z$, then $Y = Z$' (similarly).

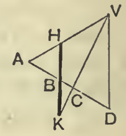
IV. Proportional division, similar figures, and the rectangle theorem (Eucl. VI. 2–18) derive from Absolute Theorem, Cor., exactly as in Ch. III., Thh. 37–53.

V. Alternando, Invertendo.
If $X : Y = Z : W$; then rect. $XW = YZ = ZY$ (Ch. III., Th. 52);
$\therefore$ $X : Z = Y : W$. (Alternando.)
Also $YZ = XW$; $\therefore$ $Y : X = W : Z$. (Invertendo.)



VI. Summation.—If $X_1 : Y_1 = X_2 : Y_2 = \&c.$,
set off $X_1, X_2, X_3 \ldots, Y_1, Y_2, Y_3 \ldots$ along the sides of an angle O, join corresp. points of divn. AP, $BQ \ldots$;
then $X_1 : X_2 = Y_1 : Y_2$, Alternando;
$\therefore$ $AP \parallel BQ \parallel CR$ (simly.), &c.;
$\therefore$ $(X_1 + X_2 + X_3 \ldots) : X_1 = (Y_1 + Y_2 + Y_3 \ldots) : Y_1$ (Prop. divn.);
$\therefore$ $(X_1 + X_2 + X_3 \ldots) : (Y_1 + Y_2 + Y_3 \ldots) = X_1 : Y_1 = \&c.$
Similarly **componendo, dividendo, ex aequali.**

To make this chapter 'pure geometry'—i.e. geometry of figure only—the only alterations required in the text are to replace the definitions of cross ratio and harmonic division by geometrical ones, and modify the proofs of Thh. 4, 6, 7, 8, 5, 27.

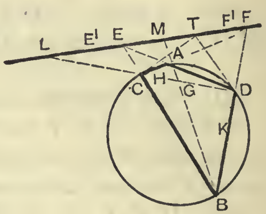


Definition.—'The **cross ratio of a pencil** V(ABCD) is HB : KB, where HK || VD.'

'The **cross ratio of a range** (ABCD) is that of any pencil on it V(ABCD).' *

The fundamental theorems follow as in text.

'A **pencil is harmonic** when the interchange of two conjugates leaves its cross ratio unaltered.' (HB : KB = KB : HB, and HB = BK.)



(**A**) Th. 4. EHGK is a transversal of quadl. ACBDEF;

∴ (FDKB) = G(FDKB) = (FCHA), on FC,
= E(FCHA) = (FBKD), on FB;

i.e. (FDKB) is unaltered on interchanging conjs. D, B.

∴ (FDKB) is harmonic.

(**B**) Thh. 6, 7, 8. Polar properties.

If CD is a fixed chd., AB any chd. of circ. or conic, through fixed point G; tangents CT, TD and quadl. ACBD form inhexagon CCADDB; ∴ intersections of opp. sides

CC, DD, i.e. T
CA, DB, i.e. F } are collinear (Pascal, proved as in text);
AD, BC, i.e. E

i.e. T, a fixed point, lies in EF, which also contains L, a fixed point (harm. conj. of fixed point G to fixed points C, D).

∴ EF is a fixed line, cutting AGB harmy. at M;

∴ polar of G, locus of M, is a str. line.

(**C**) Th. 5. Make MAB a diam. of a circle, and use polar and tangents of M; results follow easily.

(**D**) Th. 27. If GEF, GE′F′ are s.p. trs. of G, the joins of vertices EF to a fixed point C on curve determine chds. AB, A′B′, &c. through G (Constr. 2).

∴ C(EF, E′F′...) and ∴ G(EF, E′F′...) is in involn. (Th. 32).

* Just the inverse order of our Defs. 4, 5 above.

The definition of ratio and the properties of proportional lines given above are sufficient for all pure geometry—i.e. for the geometry of figure; and the tendency of modern geometry as treated by Townsend, Chasles, Cremona, and others has been to make it entirely a matter of figure, independent of number. The method, however, is only complete if and when we succeed in defining ratio or deriving the theorems of proportion entirely by figure, as we have just done.

Now that this is done, it becomes possible for every truth of geometry, whether originally stated and arrived at by extraneous aid—e.g. that of number—or not, to be expressed as some property of a figure; and '**no truth of geometry arrived at by extraneous aid can tell us about a figure any fact which cannot be arrived at by geometry only.**'

This is very well illustrated by our two treatments of cross ratio.

In Def. 4 we defined the cross ratio (ABCD) as the ratio of rectangles AB . CD : AD . CB. Then, in order to reduce this to the ratio of two magnitudes of another kind (**lines** HB : KB), we require a **nexus** associating ratios of different kinds of magnitude. This nexus is number.* But we could get no property of **figure** out of our definition until we had effected this reduction.

Similarly with the alternative definition (product of ratios), we can get no property of **figure** until we reduce to a simple ratio.

By using our second definition (the ratio HB : BK), we have just shown that all results flow directly from the definition; and it is obvious, on comparing the two processes, that that of pure geometry is the more elegant.

This does not, however, make these numerical processes useless; they often guide us to the right direction in which to look for a geometrical solution of a problem. What we ought to do in any case where we find some geometrical truth by, say, the ratio of two areas, is to look for the geometrical solution.

Treated entirely geometrically, Euclidean geometry is a complete self-contained science.

* Euclid's critical word in defining ratio is πηλικοτῆτα—'how-many-times-ness'—and he no doubt regarded ratio as a 'quotient,' though number was not completely understood in his day.

EXAMPLES—XLIII.

1. Having regard to sign, show that if A, B, C are points in a straight line in any order, $AB+BC=AC$. Extend the theorem to n points.

2. Show that, with proper signs, the sum of projections of $n-1$ consecutive sides of an n-sided polygon is equal to that of the nth side.

3. If C is a point in a straight line AB such that $AC+BC=0$, and P is any other point whatever in the line of AB, show that $AP+BP=2CP$.

4. If AB, A′B′ are any two parts of a given line, M, M′ their mid points, then $MM'=$ either $\dfrac{AA'+BB'}{2}$ or $\dfrac{AB'+BA'}{2}$.

5. If μ, ν are any two real numbers, O a point in a line AB such that $\mu AO+\nu BO=0$, and P any point whatever in the line of AB, show that $\mu AP+\nu BP=(\mu+\nu)OP$. Explain the result when $\mu+\nu$ is zero.

6. If the circumcircle of a triangle inverts about a vertex into a straight line through its own centre, find the constant of inversion, and construct a square equal to it.

7. With a vertex of a triangle and the square on a tangent from it to the incircle as vertex and constant of inversion, the incircle inverts into itself. Find the inverses of the opposite side and ecircle.

8. Any two circles are inverse figures about each centre of similitude. Prove when the circles cut.

9. If two circles are inverse figures, the centre of inversion is a centre of similitude. (Use the common tangents.)

10. Any two circles can be inverted into themselves. What is the locus of the centre of inversion? Extend the theorem to three circles.

11. Any two circles can be inverted into equal circles.

12. The angle of two lines or curves at a common point is equal to that of their inverses at the inverse point.

13. Inscribe in a circle a triangle whose sides pass through three given points.

14. Given three points A, B, C in a harmonic range, construct the fourth by ruler and pencil.

15. If AB, CD, EF are diagonals of a complete quadrilateral, A, C, E collinear, and AB, CD meet in P, and EP cuts the side CBF in K, then $(CKBF)=(CFBK)$.

16. Construct two points C, D in a line AB such that (ABCD) is harmonic, and CD is bisected at a given point M in the line AB produced. $(MB \,.\, MA=MC^2=MD^2.)$

17. If one pair of conjugates of a harmonic pencil is right-angled, it bisects the angles of the other pair.

18. The diagonals of a parallelogram are harmonic conjugates of parallels to the sides at the diagonal point. (Project into complete quadl.)

19. If M, M' and N, N' are harmonic conjugates of A, C and B, D in a parallelogram $ABCD$, parallels AP, AQ to MN, $M'N'$ form a harmonic pencil with AB, AD. (Project $M'N'$ to inf.)

20. The internal and external bisectors of an angle form a harmonic pencil with its sides.

21. If the line $AISE$ contains in- and e-centres I, E of a triangle ABC, and meets BC in S, then $(AISE)$ is harmonic.

22. In a triangle ABC, the points S, D, X, P, P' on BC are foot of bisector of A, foot of perpendicular, mid point, and points of contact of in- and e-circle. Show that:

(i.) $(DPSP')$ is harmonic, and $XS \cdot XD = XP^2$.

(ii.) If ST is the second tangent from S to the incircle, and XT meets the incircle in Y, then Y is on the nine-point circle, and these circles touch at Y. (Invert from X, constant XP^2.)

(iii.) The nine-point circle touches the ecircles.

23. Construct a tangent to a given circle from an outside point by ruler and pencil only.

24. Construct the polar and tangents to a circle of $\frac{3}{4}''$ radius from a point $1\frac{1}{2}''$ from the centre.

25. Find a point $1\frac{1}{4}''$ from the centre in Ex. 24, forming with the given point a side of a self-polar triangle, and complete the triangle.

26. Construct a self-polar triangle to a circle, $\frac{1}{2}''$ rad., one vertex $1\frac{1}{4}''$ from centre, one side from this vertex $1\frac{3}{4}''$. Measure the sides.

27. Show that every self-polar triangle of a circle is obtuse. What point is the orthocentre?

28. Given an obtuse triangle, construct its polar circle. (To which the triangle is self-polar.) What point of the triangle is the centre?

29. Construct the polar circle of a triangle, $1\frac{3}{4}''$, $1\frac{1}{4}''$, $1''$ sides. Measure the radius.

30. The polars of a point to a system of coaxial circles are concurrent.

31. If AB, BC in order in a line are $1''$, $\frac{1}{2}''$, construct D so that $(ABCD) = -\frac{2}{7}$. Measure CD.

32. If two sides of a variable triangle pass through fixed points, and the vertices move on three concurrent lines, the third side passes through a fixed point.

33. If two vertices of a variable triangle lie on fixed lines, and the sides pass through three collinear points, the third vertex lies on a fixed line.

34. The cross ratio P(ABCD) of the pencil of four fixed points on a circle at a fifth point P on it, is the ratio of the rectangles of chords AB . CD : AD . CB.

35. If O, A, B are fixed points in a fixed line, and OPQR is any transversal of three concentric circles, centre O, show that there is one point C on AB such that PA, QB, RC are always concurrent.

36. If E, F are points on opposite sides AB, CD of a quadrilateral, the intersections of AF, DE and BF, CE are collinear with the diagonal point.

37. If two triangles in a circle have fixed bases BC, EF, the line joining the intersections of their other sides AB, DE and AC, DF traverses a fixed point. (Pascal's theorem.)

38. If sides AB, CD of a quadrilateral in a conic meet in P; AD, BC in Q; AC, BD in R; show that PQR is a self-polar triangle.

39. Mark any four points A, B, C, D. Find a point P on any line through A such that P(ABCD) = a given value μ. (Constr. 4.)

40. The vertex of perspective of two figures is its own perspective.

41. Project any two triangles in perspective into similarly situated triangles. (Project axis to infinity.)

42. Project a complete quadrilateral (i.) into a rhombus, (ii.) into a square. (Constr. 6, ii.)

43. Project a hexagon in a circle into a hexagon with opposite sides parallel. (Project Pascal line to infinity.)

44. Two circles are in perspective to a centre of similitude and their radical axis.

45. Rays AP, BQ, CR from the vertices of a triangle to the opposite sides are concurrent. If angles at P, Q project into right angles, show that R projects into one also. (Property of orthocentre.)

46. A transversal cuts a system of coaxial circles in involution.

47. If coaxial circles meet in imaginary points, show that the involution on the line of centres is separate and the double points real. What circles do these points represent?

48. Represent the distances of the imaginary common points of two non-intersecting circles from the foot of their radical axis. (Constr. 7.)

49. A straight line is cut in involution by the pairs of opposite sides and the diagonals AC, BD of a quadrilateral ABCD. (See Th. 36.)

50. Rays from a point on a circle to the ends of diameters form a pencil in involution.

51. Show that if the s.p. involution of a point to a circle is right-angled, that point is the centre.

52. Construct the right-angled pair and the double rays of the s.p. involution of a point 1″ from the centre of a circle of $\frac{1}{2}$″ radius.

CHAPTER VIII.

CONICS.

FOCAL, CONJUGATE DIAMETER, POLAR, INVOLUTION, AND CROSS RATIO PROPERTIES.

Definition 1.—A **conic** is a plane curve whose radii from a fixed point, the **focus,** have a constant ratio, the **eccentricity,** to the corresponding distances from a fixed line, the **directrix.***

The **focal radius** of a point in the plane of a conic is the join of the focus F to the point.

The **axis** is the line through the focus perpendicular to the directrix DX.

Cor.—'A conic is symmetrical about its axis.'

The **vertices** are the two points A, A′ of the conic on the axis; and the **major** or **transverse axis** is the length AA′.

The **centre** is the mid point C of the major or transverse axis.

A **central** conic is an ellipse or hyperbola.

Definition 2.—A conic is **ellipse, parabola,** or **hyperbola** according as eccentricity $e<1$, $e=1$, $e>1$.

(The figures are given on the next page.)

There are two vertices A, A′ dividing FX internally and externally in the ratio $FA : AX = FA' : XA' = FP : PD = e$.

In the **ellipse,** $FA' < XA'$, $\therefore$ A, A′ are on same side of X.

In the **parabola,** A′ bisects XF externally at infinity; hence also the centre of a parabola is at infinity.

In the **hyperbola,** $FA' > XA'$, $\therefore$ A, A′ are on opp. sides of X.

Definition 3.—The **ordinate** of a point P of a conic is the perpendicular PN to the axis; the **abscissa** of P is the distance AN of the ordinate from vertex A.

The **ordinate** MP of a **diameter** QM is the parallel to the tangent at Q. QM is its **abscissa.**

Definition 4.—Chord, secant or **transversal, tangent,** and **polar** are defined as for the circle. The definitions are general.

* Some examples for plotting are given in Ch. V.

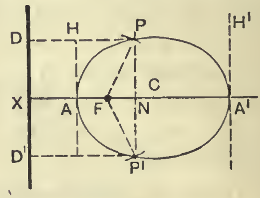
Construction 1.—'Construct a conic of given eccentricity from focus and directrix, by chords perpendicular to the axis; and determine its form.'

If $NP \perp$ axis FA, draw arcs, centre F, rad. eNX, cutting NP in P, P'; and make PD perp. to drx.

$\therefore$ $FP : PD = FP : NX = e$; similarly $FP' : P'D' = e$;

$\therefore$ P, P' are on curve, and no other point of NP is on it.

Note. P, P' move up to coincidence at A, A'; hence AH, $A'H'$ perp. to axis are tangents at the vertices.



Form of the ellipse; $e < 1$.

A, A' are on same side of X; $FP = eXN$,

$\therefore$ NP meets curve when $FN \ngtr eNX$ or eXN;

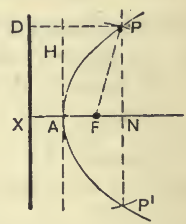
i.e. when only $FN : NX \ngtr e \ngtr FA : AX$,

or $FN : XN \ngtr FA' : XA'$;

i.e. when only N lies between A and A'.

Also, $FP < XN$, and is always finite; hence:

'The ellipse is a closed curve, lying entirely between the tangents at the vertices.'



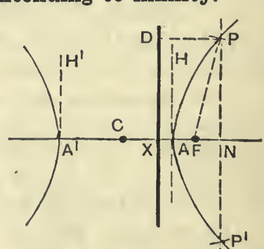
Form of the parabola; $e = 1$.

A' is at infinity, $FP = XN$,

$\therefore$ NP meets curve when $FN \ngtr NX$ or XN;

i.e. when only N lies between A and infinity on the same side as F; hence:

'The parabola is an open curve lying entirely on the focus side of the tangent at the vertex, and extending to infinity.'



Form of the hyperbola; $e > 1$.

A, A' are on opp. sides of X;

$FP = eXN$,

$\therefore$ NP meets curve when $FN \ngtr eNX$ or eXN;

i.e. when only N lies between A and inf. on the focus side of A, or between A' and inf. on the opp. side of A'; hence:

'The hyperbola is a curve of two open branches, lying entirely on opposite sides of tangents at the vertices, and extending to infinity.'

Ex. Draw (sqd. paper) the curves; $e = \frac{5}{3}$, $e = 1$, $e = \frac{3}{7}$.

Definition 5.—A **focal chord** is a chord through the focus.

The **latus rectum** is the focal chord perpendicular to the axis.

A **diameter** is a central chord; and a **diameter of a parabola** is a parallel to the axis.

The **asymptotes of a hyperbola** are the tangents from the centre to the curve.

The **subtangent** of a point on a conic is the axial intercept between the tangent and ordinate of the point.

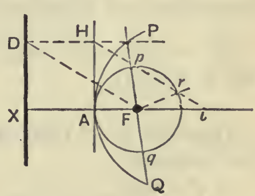
Definition 6.—A **normal** is a perpendicular to a tangent at its point of contact.

The **subnormal** of a point on a conic is the axial intercept between the normal and ordinate of the point.

Definition 7.—The **axcircle** or **auxiliary circle** of a central conic is the circle on the major or transverse axis as diameter.

The **focircle** of a conic is the circle through the inner vertex, with the focus as centre.

Construction 2.—'Construct the points on any focal ray of a conic of given eccentricity, from focus and directrix.'



Draw focircle cutting ray FP in p, q; make Fi eql. to AX, AH tangt. at vertex, meeting ip in H, make DHP perp. to drx.

Then FD = and ‖ iH, ∵ Fi = and ‖ DH;

∴ FP : PD = Fp : HD = FA : AX = e.

∴ P is on the curve.

Similarly, a second point Q on the other side of F is on the curve, and it is easily seen that no third point of FP is on the curve.

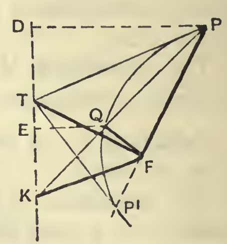
Note. DP meets Fr in one* second point R on the curve, so that every parallel to the axis meets the curve in two points P, R (which may be imaginary or coincident).

In the parabola, i is on the focircle, r coincides with i, and R is at infinity on the axis; so that DP meets the parabola in one finite point only. Hence:

Cor.—'A diameter of a parabola meets the curve in one finite point and one point at infinity.'

* DF bisects suppt. of ang. PFR, by next theorem.

Theorem 1.—'The focal radius of the directrix point of a transversal of a conic bisects the supplement of the focal angle of its chord; and the focal angle of a tangent from a point on the directrix is a right angle.'



If a transvl. PQK cuts the drx. in K, and PD, QE $\perp$ drx.; then

(i.) KP : KQ = PD : QE = FP : FQ;

$\therefore$ KF bisects an ang. F of tr. FQP;

i.e. KF bisects suppt. of focal ang. QFP.

(ii.) If Q moves to coincidence with P, KP coincides with tangt. TP,

the focal ang. QFP becomes zero, and its suppt. two rt. angs.;

$\therefore$ TF $\perp$ FP, and focal ang. TFP is a rt. ang.

Cor. (i.).—'A point P on a conic has one only tangent.'

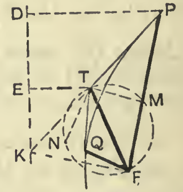
Cor. (ii.).—'Tangents at the ends of a focal chord intersect on the directrix.'

Cor. (iii.).—'A transversal meets a conic in two only points.'

Note. In the hyperbola, when P, Q are on different branches, KF bisects the focal angle itself PFQ. (Draw fig. and verify.)

Ex. Show that the tangent at the point of infinity of a parabola is the line at infinity.

Theorem 2.—'Tangents from any point to a conic have equal focal angles.'



If TP, TQ are tangts. from T to conic,

draw TM, TN perp. to FP, FQ,

draw PTK to drx., and TE, PD perp. to drx.;

then KF $\perp$ FP $\parallel$ TM;

$\therefore$ FM : FP = KT : KP = TE : PD;

i.e. FM : TE = FP : PD = e;

$\therefore$ FM = e . TE = similarly FN.

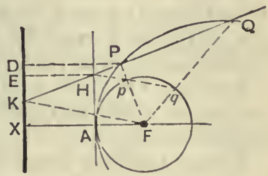
$\therefore$ rt. tr. FTM $\equiv$ FTN;

$\therefore$ ang. TFP = TFQ.

Cor.—Since any second tangt. TQ from T makes ang. TFQ = TFP, **'a point has two only tangents to a conic.'**

Construction 3.—'Construct the points in which a transversal meets a conic of given eccentricity, from focus and directrix.'

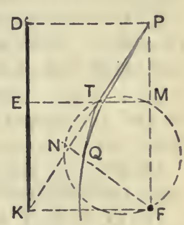
If transvl. KH meets drx. and tangt. at vertex in K, H,
draw Hpq to the focircle, parl. to KF;
join Fp, Fq to meet KH in P, Q,
and make HE, PD perp. to drx.
$\therefore$ FP : Fp = KP : KH = PD : HE;
i.e. FP : PD = Fp : HE = FA : AX = e.
$\therefore$ P, and similarly Q, is on the curve.



Note. If Hp parl. to KF meets the circle in imaginary points, KH meets the conic in imaginary points, so that a straight line meets a conic in two real, two coincident (as a tangent), or two imaginary points.

Construction 4.—'Construct the tangents from a point to a conic of given eccentricity, from focus and directrix.'

If T is the point, draw circ., diam. FT, draw TE perp. to drx., make chds. FM, FN each eql. to e . TE, make FK perp. to FM, draw KTP to P on FM.
$\therefore$ FP : FM = KP : KT = PD : TE;
$\therefore$ FP : PD = FM : TE = e.
$\therefore$ P is on the curve, and KP its tangent.
If FN meets curve in Q, ang. TFQ = TFP;
$\therefore$ TQ is the other tangent from T.



Note. The chords FM, FN can or cannot be drawn in the circle according as FT $\gtreqless e$. TE;
i.e. according as T is or is not between F and the curve.

The part of the plane on the same side of the curve as F is the **inside**, from points of which tangents cannot be drawn; and the other part the **outside** of the curve.

The second branch of the hyperbola has a second focus beyond A′, so that the outside of the hyperbola is the space between the two branches in which directrices and centre lie. Hence tangents can be drawn from the centre of a hyperbola to the curve.

EXAMPLES—XLIV.

CONICS.

1. Plot 6 ordinates, about $\frac{1}{4}''$ apart, and draw the conics, given $FX = 1''$, $e = 7:3$, $e = 1$, $e = 3:5$.

2. With the data of Ex. 1, find the points of the conics on focal rays making angles of 30°, 60°, 90°, 120° with the axis, and draw the curves.

3. Given $FX = 1{\cdot}6''$, $e = 8:7$, $e = 1$, $e = 7:9$, make XD on directrix $1''$, and angles XDP, XDQ, 90°, 108°. Construct the near point of DP, DQ on each curve.

4. One end of a string is fixed on a drawing-board, and the other to the long arm of a T-square, whose short arm moves along one edge of the board. Find the locus of a point which keeps the string tight against the long arm. Construct a parabola in this way.

5. On a diagonal FD from a fixed point F to a fixed straight line a rhombus FPDQ is described, FQ having a fixed direction. Find the locus of P. Can it be an ellipse? Why?

6. The vertex A of a triangle is fixed, B describes a fixed line, BC has a given direction. Find the locus of C when AC : BC is constant.

7. Plot 4 points on the locus of Ex. 6, dist. from A to line 1 cm., ang. of BC to line 60°, $AC : BC = 3:2$. (Draw CD perp. to line, calculate AC : CD.)

8. A circle touches a given circle and a given line. Find the locus of its centre. Find also the locus of the centre of a circle touching the arc and chord of a given segment of a circle.

9. A circle, centre F, cuts a conic in R, R′. A chord PQ cuts RR′ in H and the directrix in K; Hqp parallel to KF cuts FQ, FP in q, p. Show that q, p are on the circle. (Draw HE, PD, RM perp. to drx.)

10. P is any point, Q, R fixed points, on a conic. The chords PQ, PR meet the directrix in H, K. Show that the focal angle HFK is constant.

11. Q, R, S, T are four fixed points, P any point, on a conic. The pencil P(QRST) cuts the directrix in HKLM; show that the pencil F(HKLM) is of given form—i.e. has given angles in order.

12. Two fixed tangents AT, BT determine an intercept AB on a movable tangent AA′. Show that the focal angle AFB is constant.

13. A movable tangent cuts four fixed tangents in A, B, C, D. Show that the pencil F(ABCD) has constant form.

14. A focal chord of a conic is harmonically divided by focus and directrix.

15. The semi-latus-rectum is the harmonic mean of the parts of a focal chord. (If PFP′ cuts drx. in K, and PD, P′D′ $\perp$ drx., FX is H.M. of PD, P′D′ by Ex. 16.)

16. Show that the directrix is the polar of the focus. Also that the polar of any point on the directrix is a focal ray.

17. Deduce the cross-ratio properties of a conic from Exx. 11, 13.

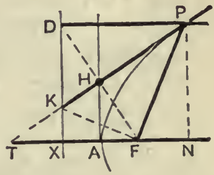
Theorem 3.—'Of any point on a parabola,'

(i.) 'The tangent bisects the angle of the focal radius and diameter;'

(ii.) 'The focal radius is equal to the axial intercept of focal radius and tangent;'

(iii.) 'The subtangent is equal to twice the abscissa;'

(iv.) 'The foot of the focal perpendicular on a tangent lies on the tangent at the vertex.'



If tangt. at P meets axis in T, and drx. in K, and if PD ⊥ DX, FH ⊥ PT; then

(i.) KFP = rt. ang. = KDP (Th. 1), and FP = PD, ∵ $e = 1$;

∴ rt. tr. FPK ≡ DPK.

∴ tangt. TP bisects ang. FPD of focal rad. FP and diam. DP.

(ii.) Ang. FTP = alt. ang. TPD = FPT;

∴ focal rad. FP = axial intercept FT.

(iii.) TF = FP = DP = XN,

∴ TX = FN, and TA = AN;

i.e. subtangt. TN = 2 . AN.

(iv.) KP is rt. bisr. of FD,

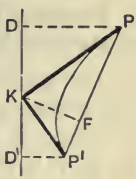
∴ H is mid point of FD;

∴ HA || DX ⊥ FX;

i.e. H is on the tangt. at vertex A.

Cor.—'Two tangents to a parabola cannot be parallel.'

Theorem 4.—'Tangents to a parabola at the ends of a focal chord intersect at right angles on the directrix.'



If PFP′ is a focal chord, and FK ⊥ PP′, then the tangents at P, P′ meet FK on drx. DD′ in K, say.

Also, PK, P′K bisect angs. FKD, FKD′;

∴ PK ⊥ P′K.

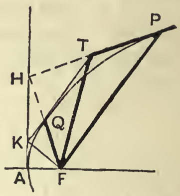
Ex. 1. Show that the circumcircle of FHP, in Th. 3, touches AH.

Ex. 2. Perpendicular tangents to a parabola intersect on the directrix. (Converse of Th. 4.)

Ex. 3. Show that K bisects DD′ in Th. 4.

Theorem 5.—'The focal triangles of two tangents from a point to a parabola are similar.'

If TP, TQ are tangts. to a parabola, meeting tangt. of vertex at H, K;
then angs. A, FKQ, FHT are rt. angs.,
and ang. KFQ = KFA, at focus;
∴ tr. FQK ||| FKA.
∴ ang. FQK = FKA = suppt. of FKH
= FTH, in cyclic quadl. FKHT;
∴ ang. FQT = suppt. of FQK = FTP.
Also, ang. QFT = TFP, in trs. QFT, TFP;
∴ tr. FQT ||| FTP.

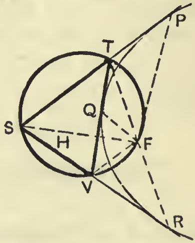


Ex. Show that the triangle FTP can be derived from FQT by rotation and multiplication. What is the multiplier?
Show also that △FTP : △FQT = FP : FQ.

Note. This Ex. helps us to remember which angles are equal.

Theorem 6.—'The circumcircle of a circumscribing triangle of a parabola passes through the focus.'

If tr. STV circumscribes a parabola, and P, Q, R are points of contact;
then tr. FQT ||| FTP;
and FRS ||| FSP;
∴ ang. FTV = FPS
= FSV;
∴ F lies on circle STV.



Theorem 7.—'The orthocentre of a circumscribing triangle of a parabola lies on the directrix.' (Steiner.)

The tangent at the vertex, containing feet of perps. from F on the tangts., is the Simson line of F in triangle STV (last fig.).
∴ tangt. at vertex bisects the join FH to the orthocentre. (Ex. iv. p. 112, Ch. IV.)
∴ H is on the directrix.
This may also be derived from Brianchon's theorem.

Ex. Construct a circumtriangle, given in position the orthocentre and two sides tangent to the parabola.

Theorem 8.—'In a parabola,'

(i.) 'The axial intercept of the focal radius and normal of a point is equal to the focal radius;' (FG = FP);

(ii.) 'The subnormal is equal to the semi-latus-rectum;' (NG = 2AF);

(iii.) 'If PN is the ordinate of P, then $PN^2 = 4AF \,.\, AN$.'

If PT, PG, DPE are tangt., normal, and diam. of P; then

(i.) TP bisects ang. FPD,

∴ PG, perp. to TP, bisects suppt. FPE.

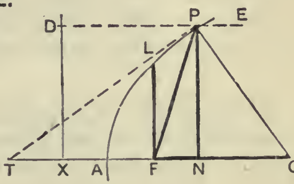
∴ ang. FPG = EPG = FGP;

∴ FG = FP.

(ii.) FG = FP = DP = XN;

∴ NG = XF = FL = 2AF.

(iii.) $\dfrac{PN}{TN} = \dfrac{NG}{PN}$ (sim. trs.) $= \dfrac{2AF}{PN}$;



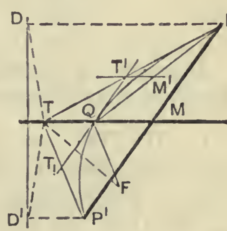
∴ $PN^2 = 2AF \,.\, TN = 4AF \,.\, AN$.

Ex. Show that the curve $PN^2 = 4AF \,.\, AN$ (A and AF fixed and $PN \perp AF$), is a parabola.

Theorem 9.—(i.) 'A diameter of a parabola bisects chords parallel to its tangent; and the curve bisects the diametral distance of a point from the chord of contact of its tangents.'

(ii.) 'If PM is an ordinate of a diameter QM, then $MP^2 = 4FQ \,.\, QM$.'

If TP, TP′ are tangts. of chd. PP′, $T'QT_1$ tangt. of Q, and TQM, T′M′, DP, and D′P′ diams.; then,



(i.) tr. DPT ≡ FPT,

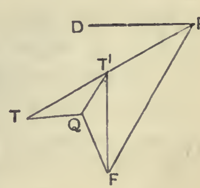
∴ TD = TF = simly. TD′;

∴ TM is rt. bisr. of DD′ and bisects PP′;

simly. T′M′ bisects PQ, PT; T_1 bisects TP′;

∴ $T'QT_1 \parallel PP'$, and bisects TM;

i.e. diam. QM bisects chd. of P parl. to QT′, and Q bisects TM.



(ii.) Ang. FQT′ = TQT′; and tr. FQT′ ||| FT′P;

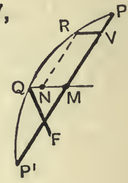
∴ ang. FT′Q = FPT′ = DPT = T′TQ;

∴ tr. FQT′ ||| T′QT;

∴ $\dfrac{QT'}{FQ} = \dfrac{TQ}{QT'}$; i.e. $QT'^2 = FQ \,.\, TQ = FQ \,.\, QM$;

∴ $MP^2 = 4QT'^2 = 4FQ \,.\, QM$.

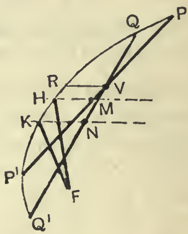
Theorem 10.—'The rectangle of the parts of any transversal from a point to a parabola is four times the rectangle of the focal radius of its conjugate diameter and its diametral distance from the curve.' $(\mathsf{VP}\,.\,\mathsf{VP}' = 4\mathsf{FQ}\,.\,\mathsf{VR}.)$



If RV, PVP' are the diam. dist. and transvl. of V, and QM the diam. conj. to PP'; make RN parl. to PP'.

$$\therefore\ \mathsf{VP}\,.\,\mathsf{VP}' = \mathsf{MV}^2 - \mathsf{MP}^2 = \mathsf{NR}^2 - \mathsf{MP}^2$$
$$= 4\mathsf{FQ}\,.\,\mathsf{QN} - 4\,.\,\mathsf{FQ}\,.\,\mathsf{QM}$$
$$= 4\mathsf{FQ}\,.\,\mathsf{MN} = 4\mathsf{FQ}\,.\,\mathsf{VR}.$$

Theorem 11.—Apollonius' theorem: 'The ratio of the rectangles of parts of two transversals of given directions from any point to a parabola is constant, and is equal to that of the focal radii of their conjugate diameters.'

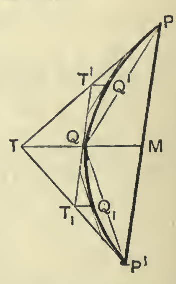


If PVP', QVQ' are transvls. of V in fixed directions, their conj. diams. HM, KN, and points H, K are fixed.

Hence, if VR is diam. dist. of V,

$$\therefore\ \frac{\mathsf{VP}\,.\,\mathsf{VP}'}{\mathsf{VQ}\,.\,\mathsf{VQ}'} = \frac{4\,.\,\mathsf{FH}\,.\,\mathsf{VR}}{4\,.\,\mathsf{FK}\,.\,\mathsf{VR}}$$
$$= \frac{\mathsf{FH}}{\mathsf{FK}} = \text{constant.}$$

Theorem 12.—'The area of a segment of a parabola is two-thirds of that of the triangle of its chord and tangents.'



If TP, TP' are tangts. of chd. PP', and TQM the diam. of T, draw tangt. $\mathsf{T'QT_1}$, and diams. $\mathsf{T'Q'}$, $\mathsf{T_1Q_1}$; and repeat the process indefinitely.

Then $\triangle\,\mathsf{PQM} = \triangle\,\mathsf{PTQ} = 2\,\triangle\,\mathsf{TQT'}$;
similarly, $\triangle\,\mathsf{P'QM} = 2\,\triangle\,\mathsf{TQT_1}$,
and $\triangle\,\mathsf{PQP'} = 2\,\triangle\,\mathsf{TT'T_1}$.

Similarly, area of inpoln. $\mathsf{PQ'QQ_1P'} = 2$ area of corresp. circumpoln.; and so on.

And if the process is infinitely continued, the inner perimeters of circum- and in-polygons coincide with the curve;

$\therefore$ area of segment $\mathsf{PQP'} = 2(\triangle\,\mathsf{TPP'} - \text{segt. } \mathsf{PQP'}) = \frac{2}{3}\,\triangle\,\mathsf{PTP'}$.

EXAMPLES—XLV.

The Parabola.

1. Find the locus of the centre of a circle which passes through a given point and touches a given line.

2. Focal radii of a parabola increase with their angle from FA.

3. The shortest focal chord of a parabola is the latus rectum.

4. A circle on a focal chord as diameter touches the directrix.

5. The radius of the circle of Ex. 4 through the point of contact of the directrix meets the curve in Q. Show that the radius is parallel to the axis, and equal to 2 . FQ.

6. Find the envelope of one side of a right angle whose vertex moves on a line, and whose other side passes through a fixed point. (See Ch. V., 'Plotting Loci.') Generalise this.

7. Construct the tangent and normal of a point P on a parabola by the circle, centre F, radius FP.

8. Draw a parabola of given focus to touch a given line at a given point.

9. Given focus, directrix, and the line of a tangent to a parabola, construct the point of contact.

10. If PG, PD are normal and perpendicular to directrix of a point P on a parabola, FP bisects DG.

11. If PT is the tangent of P in Ex. 10, PT and FD right-bisect each other. What figure is FPDT? FGPD?

12. The tangent from A to a circle on a focal radius FP as diameter is half the ordinate of P.

13. The envelope of the fold of a rectangle of which one vertex is made to travel along an opposite side, is a parabola.

14. If Q is the point on a parabola of the conjugate diameter of a focal chord PP′, show that $PP'=4FQ$.

15. If TP, TQ are tangents to a parabola, $FT^2=FQ . FP$.

16. The intersections of the joins of the ends, two and two, of two parallel chords of a parabola lie on the conjugate diameter.

17. If R, S are the points of intersection of Ex. 16, show that the curve bisects RS. (Use polar.)

18. The locus of mid points of focal chords of a parabola is a parabola of half its dimensions.

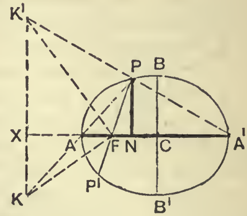
19. The rectangle FP . FP′ of a focal chord is equal to FA . PP′.

20. A focal chord PP′ and its rectangle FP . FP′ increase with the angle made by the larger part with the latus rectum.

21. If a circle cuts a parabola in P, P′, Q, Q′, show that PP′ and QQ′ make equal angles with the axis.

22. If VPP′, VQQ′, two transversals from a point V to a parabola, meet the axis in R, S, then $VP . VP' : VQ . VQ' = VR^2 : VS^2$.

Theorem 13.—'If NP is an ordinate of an $\substack{\text{ellipse}\\ \text{hyperbola}}$' the ratio $\dfrac{\mathsf{NP}^2}{\mathsf{NA}\,.\,\mathsf{NA}'}$ is constant and equal to $\substack{-b^2:a^2\\ +b^2:a^2}$, where $2a$, $2b$ are the axes AA′, BB′.'



Draw PAK, PA′K′ to drx., join KF, K′F; then KF ⊥ K′F (bisrs. of AFP′, AFP);

∴ XF is mean propl. of KX, XK′.

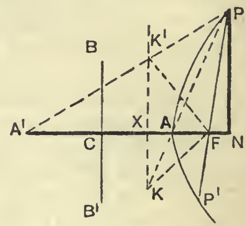
Also, $\dfrac{\mathsf{NP}}{\mathsf{NA}}=\dfrac{\mathsf{KX}}{\mathsf{AX}}$; $\dfrac{\mathsf{NP}}{\mathsf{NA}'}=\dfrac{\mathsf{XK}'}{\mathsf{XA}'}$;

$\therefore \dfrac{\mathsf{NP}^2}{\mathsf{NA}\,.\,\mathsf{NA}'}=\dfrac{\mathsf{KX}\,.\,\mathsf{XK}'}{\mathsf{AX}\,.\,\mathsf{XA}'}=\dfrac{\mathsf{XF}^2}{\mathsf{AX}\,.\,\mathsf{XA}'}=\text{const.}$

(i.) In the **ellipse**, the **minor axis** is the diam. BCB′, perp. to AA′.

$\therefore \mathsf{NP}^2:\mathsf{NA}.\mathsf{NA}'=\mathsf{CB}^2:\mathsf{CA}.\mathsf{CA}'=-b^2:a^2.$

(ii.) In the **hyperbola**, construct b so that $b^2:a^2=\mathsf{NP}^2:\mathsf{NA}\,.\,\mathsf{NA}'$, which is positive; and make $\mathsf{B'C}=\mathsf{CB}=b$; then B′CB is the **conjugate axis.**

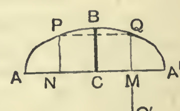


Also, if β, β' are the imaginary points in which CB meets the curve; then

$\mathsf{C}\beta^2:a^2=-\mathsf{C}\beta^2:\mathsf{CA}\,.\,\mathsf{CA}'=-b^2:a^2.$

$\therefore \mathsf{C}\beta=\mathsf{C}\beta'=b\sqrt{-1}$ or ιb.

Theorem 14.—'An $\substack{\text{ellipse}\\ \text{hyperbola}}$ is symmetrical about its $\substack{\text{minor}\\ \text{conjugate}}$ axis and has two foci; and the centre bisects all diameters.'

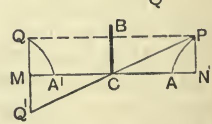


Make CM = CN, draw ordinate MQ;

∴ MA = − NA′; MA′ = − NA;

∴ MA . MA′ = NA . NA′, and $\mathsf{MQ}^2=\mathsf{MP}^2$;

∴ CB is rt. bisr. of PQ, and the curve is symmetrical about its second axis;



∴ there is a second focus F′ (CF′ = FC), and a second drx. D′X′ (CX′ = XC);

also, if QQ′ is the chd. bisected at M, PCQ′ is the diam. of P bisected at C.

Cor.—'The centre of a conic is the pole of the line at infinity.'

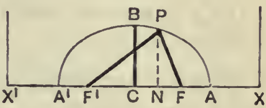
For the harm. conj. of C to P, Q′ is at inf.

Theorem 15.—'In an ellipse,'*

(i.) '$CF : CA = e = CA : CX$;'

(ii.) 'The sum of focal radii is equal to the major axis;' $(FP + F'P = A'A = 2a)$;

(iii.) '$FB = a$; $CF^2 = a^2 - b^2$; $FA \,.\, A'F = b^2$; $e^2 = (a^2 - b^2)/a^2$.'



(i.) $e = \dfrac{FA}{AX} = \dfrac{A'F}{A'X} = \dfrac{A'F \pm FA}{A'X \pm AX}$

$= \dfrac{2CA}{2CX}$ or $\dfrac{2CF}{2CA}$;

i.e. $CA : CX = e = CF : CA$.

(ii.) $FP + F'P = eNX + eX'N = eX'X$

$= A'A = 2a$.

(iii.) $FB = F'B = \frac{1}{2}(FB + F'B) = a$.

$CF^2 = FB^2 - CB^2 = a^2 - b^2$.

$FA \,.\, A'F = CA^2 - CF^2 = b^2$.

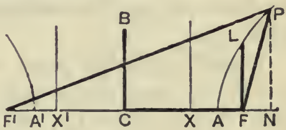
$e^2 = CF^2 : CA^2 = (a^2 - b^2) : a^2$.

Theorem 16.—'In a hyperbola,'

(i.) '$CF : CA = e = CA : CX$;'

(ii.) 'The difference of focal radii of a point is equal to the transverse axis;' $(F'P - FP = A'A = 2a)$;

(iii.) '$FL = (e^2 - 1)a$;† $e^2 = (a^2 + b^2)/a^2$; $CF^2 = a^2 + b^2 = AB^2$; $FA \,.\, FA' = b^2$.'



(i.) $e = \dfrac{AF}{XA} = \dfrac{A'F}{A'X} = \dfrac{A'F \pm AF}{A'X \pm XA}$

$= \dfrac{2CF}{2CA}$ or $\dfrac{2CA}{2CX}$;

i.e. $CF : CA = e = CA : CX$.

(ii.) $F'P - FP = e(X'N - XN) = A'A = 2a$.

(iii.) $FL^2 = e^2 XF^2 = e^2(CF - CX)^2 = e^2\left(ea - \dfrac{a}{e}\right)^2 = (e^2 - 1)^2 a^2$;

$FA \,.\, FA' = CF^2 - CA^2 = (e^2 - 1)a^2$;

$\therefore \dfrac{b^2}{a^2} = \dfrac{FL^2}{FA \,.\, FA'}$ (Th. 13) $= e^2 - 1$; $\therefore e^2 = \dfrac{a^2 + b^2}{a^2}$;

$\therefore CF^2 = e^2 a^2 = a^2 + b^2 = AB^2$; $FA \,.\, FA' = CF^2 - CA^2 = b^2$.

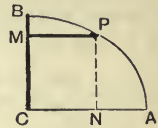
Ex. 1. The curves $FP \pm F'P =$ const. are conics.

Ex. 2. The tangent from F to axcircle of hyperbola $= b$.

* F, A, X are on right of C for convenience of *sign.*

† FL is semi-latus-rectum.

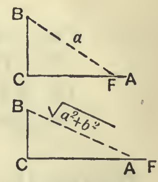
Theorem 17.—'**In an ellipse, if PM is an ordinate to BB′, the ratio $MP^2 : b^2 - CM^2 = a^2 : b^2$.**'



$$\frac{a^2}{b^2} = -\frac{NA \cdot NA'}{NP^2} = \frac{CA^2 - CN^2}{CM^2}$$
$$= \frac{CN^2}{b^2 - CM^2} \text{ (summation)} = \frac{MP^2}{b^2 - CM^2}.$$

'**In a hyperbola, $MP^2 : b^2 + CM^2 = a^2 : b^2$.**'
(Prove as above.)

Theorem 18.—'**The curve $NP^2 : NA \cdot NA' =$ constant, AA′ given and N a right angle, is an ellipse or hyperbola according as the sign of the ratio is negative or positive.**'

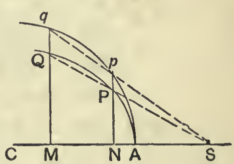


Make $a = CA = A'C$, and $b^2 : a^2 = \mp NP^2 : NA \cdot NA'$.
(i.) For the − sign;
make $CB = b$, perp. to CA, $BF = a$.
Construct ellipse, foc. F, vert. A, ecc. CF : CA.
This coincides with the given curve.

Note. If $b > a$, make b the major axis.

(ii.) For the + sign; make $CB = b$, $CF = BA = \sqrt{a^2 + b^2}$.
Construct hyperbola, foc. F, vert. A, ecc. CF : CA.
This coincides with the given curve.

Theorem 19.—'**An ellipse can be derived from its axcircle by multiplying the ordinates of the circle to the axis by the ratio $b : a$.**'



If pPN, qQM are joint ordinates of circle and ellipse from major axis;
$NP^2 : Np^2 = -NP^2 : NA \cdot NA' = b^2 : a^2$.
∴ $NP : Np = b : a =$ simly. $MQ : Mq$.
P, p or Q, q are corresp. points of the curves; also, pPN and qQM are similarly divided;
∴ qp, QP, MN are concurrent on the axis. Hence:

Cor.—'**The ellipse is the perspective of the axcircle, vertex at infinity along BB′, axis of perspective AA′.**'

The tangents at corresp. points P, p meet on the axis; and the line at inf. in either fig. is its perspective in the other.

Ex. 1. Prove the corresponding theorem, using the circle on the minor axis as diameter.

Ex. 2. Show that Thh. 13, 15, 17 for the ellipse can be written for the hyperbola by using $-b^2$ for $+b^2$.

Theorem 20.—(i.) '**If PG is the normal of an ellipse or hyperbola, FG : FP = e.**'

(ii.) '**The tangent and normal of a point of an ellipse or hyperbola are the bisectors of angle of the focal radii of the point.**'

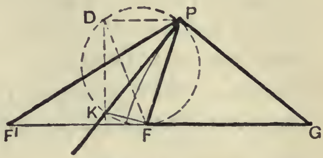


Fig. of Hyp.
(Same proof for Ell.)

If the tangt. of P meets drx. in K, and PD ⊥ drx., KFP is a rt. ang.;

∴ circ. on diam. KP passes through F, D and touches normal PG at P.

∴ ang. FPG = FDP, opp. arc;

and ang. PFG = FPD, alt. ang.,

∴ tr. FGP ||| PFD;

∴ FG : FP = FP : DP = e (without regard to sign).

Simly. F′G : F′P = e = FG : FP;

∴ PG and its perp. KP are bisectors of ang. P of tr. FPF′.

Theorem 21.—(i.) '**The foot of a focal perpendicular of a tangent of an ellipse or hyperbola lies on the axcircle.**'

(ii.) '**The rectangle of focal perpendiculars on a tangent of an ellipse or hyperbola is equal to the square on CB.**' (FM . F′M′ = $\pm b^2$.)

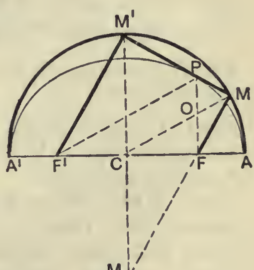


Fig. of Ell.
(Same proof for Hyp.)

If FM, F′M′ ⊥ MP the tangt. at P, then

(i.) the centre O of circle FMP is the mid point of FP;

∴ CO || F′P (∵ C bisects F′F).

Also, ang. OMP = OPM = F′PM′,

∴ OM || F′P, and O is on CM.

Also, OM = $\frac{1}{2}$FP; CO = $\frac{1}{2}$F′P;

∴ CM = $\frac{1}{2}$(FP + F′P) = a.

∴ M, and simly. M′, is on the axcircle.

(ii.) If MF meets axcircle in M_1, $M'M_1$ is a diam., and M_1F = F′M′.

∴ FM . F′M′ = M_1F . FM = A′F . FA = b^2. (Thh. 15, 16 (iii.).)

Note. The point of contact P of tangent from M on the axcircle is on the circle through F touching the axcircle at M.

We can draw tangents from any point T by drawing a circle on diam. TF to cut the axcircle in M, K, say; TM, TK are tangts.

Theorem 22.—'Tangents to an $\begin{matrix}\text{ellipse}\\ \text{hyperbola}\end{matrix}$ from a point form equal angles with the focal rays of the point.'

If TP, TQ are tangts. from T, draw perps. FM, FN, F′M′, F′N′ from foci;

$\therefore$ FM . F′M′ $= b^2 =$ FN . F′N′.

$\therefore$ FM : FN = F′N′ : F′M′.

Turn fig. TMFN over into posn. THVK;

$\therefore$ tr. VKH ||| F′N′M′, and is simly. situated about T;

$\therefore$ TVF′ is a str. line.

$\therefore$ ang. FTP = VTH = F′TQ.

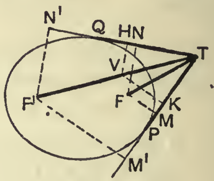


Fig. of Ell.
(Same proof for Hyp.)

Theorem 23.—'Perpendicular tangents of an $\begin{matrix}\text{ellipse}\\ \text{hyperbola}\end{matrix}$ intersect on a fixed circle.' (The **director circle.**)

If perp. tangts. TP, TQ meet the axcircle in M, M′, N, N′, the quadls. FT, F′T are rectangles.

Also, $CT^2 = TM \cdot TM' + CM^2$ (Th. 93, Ch. V.)

$= FN \cdot F'N' + CM^2$

$= a^2 + b^2$.

$\therefore$ CT is constant, and locus of T is a circle.

In the hyperbola, $CT^2 = a^2 - b^2$.

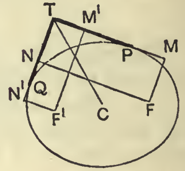
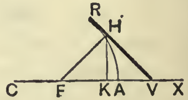


Fig. of Ell.
(Same proof for Hyp.)

Ex. Prove conversely that tangents from a point on the director circle are perpendicular.

Note. The perp. tangts. from an axial point V of the director circle, as VHR, make angs. of 45° with the axis, by symmetry.

Hence, if FH ⊥ VR, and HK ⊥ CA, KV = KH = FK. (Isosc. rt. trs.)



Thus, if C moves to infinity, F and A remaining fixed, HK coincides with tangt. at A (limiting axcircle), the conic becomes a parabola, KV coincides with AX, and the limiting director circle passes through X.

Thus the director circle of the parabola is the directrix. (Th. 4.)

Ex. Interpret Th. 18 for the parabola in the same way.

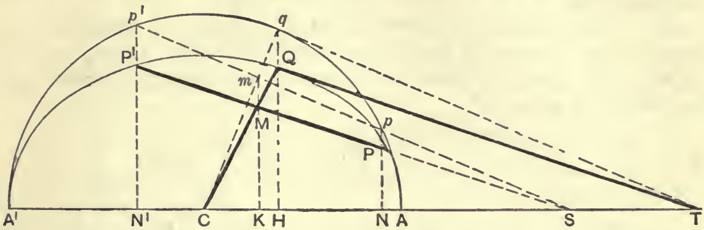
Definition 8.—Conjugate diameters are pairs of diameters each of which bisects chords parallel to the other.

Any line through a point is **conjugate** to the diameter bisecting a chord parallel to the line.

Theorem 24.—'In an ellipse,'

(i.) 'A diameter bisects chords parallel to its tangents;'

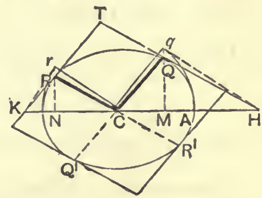
(ii.) 'Conjugate diameters correspond to perpendicular diameters of the axcircle, and their tangents form a parallelogram.'



(i.) If PP′ is a chd. parl. to tangt. QT, pp', qT corresp. chd. and tangt. of axcircle,* M, m mid points of chds.,
and PN, P′N′, QH, MK perps. to AA′; then
KM ∥ PN and P′N′, ∴ KM bisects pp' in m,
and Km : KM = Np : NP = Hq : HQ;
∴ fig. SKMm ||| THQq (∵ TQ ∥ SM);
∴ pp' ∥ Tq ⊥ Cm and Cq.
∴ C, m, q, and ∴ also C, M, Q, are collinear;
i.e. diam. CQ bisects chds. PP′ parl. to its tangt. QT.

Cor.—'Parallels of axcircle correspond to parallels of ellipse.'

(ii.) If CQ bisects chds. parl. to CR, then CR ∥ tangt. HQ of diam. CQ;
and if Cq, Cr are corresp. diams. of axcircle, fig. CrRN ||| HqQM.
∴ Cr ∥ Hq ⊥ Cq, and tangt. Kr ⊥ Cr ∥ Cq;
∴ fig. KrRN ||| CqQM, and KR ∥ CQ;
∴ CR bisects chds. parl. to CQ;



i.e. CQ, CR are conj. diams., and the corresp. diams. of axcircle Cq, Cr are perpendicular.

Also, tangts. at Q, Q′, R, R′ form a parallelogram.

Ex. Supplemental chords† are parallel to conjugate diameters.

Definition 9.—The **eccentric angle** of a point on an ellipse is the angle of the axis and the diameter of the corresponding point of the axcircle.

* Large and small letters are used for corresponding points.

† Chds. from a point on curve to the ends of a diam.

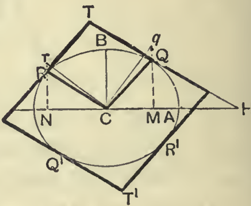
Theorem 25.—'In an ellipse,'

(i.) 'The ordinate and central abscissa of a point of eccentric angle ϕ are $b \sin\phi$, $a\cos\phi$;'

(ii.) 'The sum of squares of conjugate semi-diameters has constant area a^2+b^2;' ($CR^2+CQ^2=a^2+b^2$);

(iii.) 'The rectangle of focal radii of a point is equal to the square of its conjugate semi-diameter;' ($FQ \,.\, F'Q = CR^2$);

(iv.) 'The parallelogram of tangents of conjugate diameters has constant area $4ab$.'



If CQ, CR are conj., ϕ, ϕ' ecc. angs. of Q, R, and q, r corresp. points of axcircle, and TH, TR tangts.; then

(i.) $CM = Cq \cos\phi = a\cos\phi$;

$MQ = \frac{b}{a} \,.\, Mq = b\sin\phi$.

(ii.) $\phi' = rCH = 90° + qCH = 90° + \phi$;

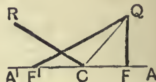
$\therefore\ CQ^2 + CR^2 = CM^2 + MQ^2 + NC^2 + NR^2$

$$= a^2\cos^2\phi + b^2\sin^2\phi + a^2\sin^2\phi + b^2\cos^2\phi$$
$$= a^2 + b^2.$$

Cor.—'$Nr = CM$; $NC = Mq$.'

(iii.) $(2a)^2 = (FQ + F'Q)^2 = FQ^2 + F'Q^2 + 2FQ \,.\, F'Q$

$$= 2CF^2 + 2CQ^2 + 2FQ \,.\, F'Q;$$



$\therefore\ 2a^2 = (a^2 - b^2) + (a^2 + b^2 - CR^2) + FQ \,.\, F'Q$;

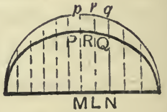
$\therefore\ FQ \,.\, F'Q = CR^2$.

(iv.) $\square\, TT' = 4 \,.\, \square\, TC = 8 \,.\, \triangle\, RCQ = 8 \,.\, \triangle\, RCH$

$$= 4 \,.\, RN \,.\, CH = 4 \,.\, b\cos\phi \,.\, CH = 4b \,.\, Cq$$
$$= 4ab.$$

Ex. The least sum of conjugate diameters is $2(a+b)$.

Theorem 26.—'The area of an ellipse is πab.'



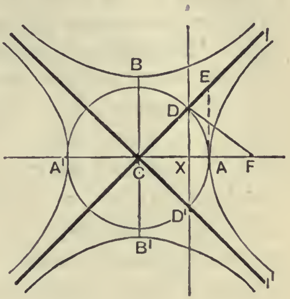
Divide the semi-ellipse and semi-axcircle by ordinates into narrow strips PN, pN; then if rRL bisects trapeziums PN, pN, area PN : pN = MN . RL : MN . rL = $b : a$.

Hence, if the lines of division PM, QN... move to coincidence, the number of strips becoming infinite, area of semi-ell. : semicirc. $= b : a$.

$\therefore$ area of ellipse $= \frac{b}{a} \times$ area of circ. $= \pi ab$.

Theorem 27.—'The asymptotes of a hyperbola touch the curve at infinity.'

Construction 5.—'Construct the asymptotes of a hyperbola; and construct the conjugate axis.'



If X is foot of drx., AE tangt. at A, and FD tangt. from F to axcircle; then tr. CDF $\equiv$ CAE.

$\therefore$ CD : CE = CA : CF = CX : CA;

$\therefore$ DX $\parallel$ AE $\perp$ AA′, and D is on drx.

Also, since D on the axcircle is foot of perp. from F on CD, CD, and simly. CD′, is a tangt. to the hyperbola;

$\therefore$ the circle through F, touching the axcircle at D, meets CD in the point of contact I. (Note, p. 233.)

But since FD is tangent to the axcircle, this circle must be the limiting circle, centre at inf. along CD, represented by FD; and it therefore cuts CD at inf.

Hence the asymptotes touch the hyperbola at infinity.

To construct the asymptotes, join C to the points D, D′ in which the axcircle cuts the directrix.

Also, $b^2 = \mathrm{FA} \,.\, \mathrm{FA}' = \mathrm{FD}^2$, $\because$ FD touches axcircle;

$\therefore$ $b = \mathrm{FD}$.

To construct the conjugate axis, join F to a point of intersection D of axcircle and directrix, then $b = \mathrm{FD}$.

If 2α is the angle of the asymptotes,

$$\tan\alpha = \frac{\mathrm{AE}}{\mathrm{CA}} = \frac{\mathrm{DF}}{\mathrm{CD}} = \frac{b}{a}; \text{ or } \alpha = \tan^{-1}\frac{b}{a}.$$

Note. Only those diameters in the angle of the asymptotes containing A meet the curve in real points. All diameters in the angle containing B meet the curve in imaginary points.

Definition 10.—The **conjugate hyperbola** is the hyperbola whose transverse axis is BB′, and conj. axis AA′.

It has the same asymptotes; and we show later that an imaginary diameter of the hyperbola in direction CQ is $\sqrt{-1}$QQ′, where QQ′ is the real diam. of the conj. hyp. in this direction.

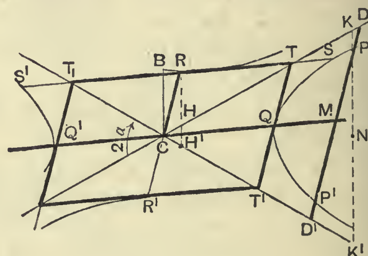
Theorem 28.—'In a hyperbola,'

(i.) 'If a chord PP' meets the asymptotes in D, D', then $PD \,.\, PD' = -CR^2$, where CR is the parallel semi-diameter of the conjugate hyperbola;'

(ii.) '$DP = P'D'$, and the asymptotal intercept of a tangent is bisected at the point of contact;'

(iii.) 'A diameter bisects chords parallel to its tangents;'

(iv.) 'Tangents of conjugate diameters form a parallelogram with the asymptotes as diagonals.'



If KPK', $RHH' \parallel CB$;

then (i.) tr. $PKD \,|||\, RHC$;

and $PK'D' \,|||\, RH'C$.

$\therefore \dfrac{PD \,.\, PD'}{PK \,.\, PK'} = \dfrac{CR \,.\, CR}{RH \,.\, RH'}$;

and $PK \,.\, PK' = NP^2 - NK^2$

$= \{(CN^2 - CA^2) - CN^2\} \tan^2\alpha$

$= -CA^2 \tan^2\alpha = -b^2$.

Simly. $RH \,.\, RH' = +b^2$,

in conj. hyp. (using Th. 17);

$\therefore PD \,.\, PD' = -CR^2 =$ simly. $P'D \,.\, P'D'$.

(ii.) $MD^2 - MP^2 = -PD \,.\, PD' = -P'D \,.\, P'D'$

$= MD'^2 - MP'^2$, if M bisects DD';

$\therefore MP = MP'$, $DP = P'D'$, M bisects chd. PP'.

Also, if PP' moves parl. to CR to coincide with tangt. TQT', then P, P' coincide with Q; and D, D' with T, T';

$\therefore TQ = QT'$, and $QT^2 = -QT \,.\, QT' = CR^2$;

$\therefore QT =$ and $\parallel CR$, hence $TR =$ and $\parallel CQ$.

(iii.) C, Q, M are collinear, $\because TQT'$, DMD' are similarly divided;

$\therefore CQ$ bisects chd. PP' parl. to its tangt.

(iv.) Draw TR to T_1; then CQ bisects T_1T',

$\therefore CR$ bisects TT_1, hence R bisects chd. STS', by (ii.),

and TRT_1 is tangt. at R to conj. hyperbola;

i.e. CR and CQ bisect chds. parl. to the other in either hyperbola, and are conj. diams. of each.

Also tangts. at Q, Q', R, R' meet on asms. and form a parm.

Note. If CR meets the first hyp. in ρ, ρ'; then $\rho\rho'$ meets asms. in C, C.

$\therefore C\rho^2 = \rho C \,.\, \rho C = PD \,.\, PD' = -CR^2$; and $C\rho = \sqrt{-1} \,.\, CR$.

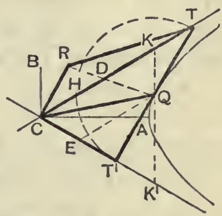
We may use without confusion $C\rho$ or CR as the conj. semi-diam. of CQ.

Theorem 29.—'In a hyperbola,'

(i.) 'The triangle of a tangent and the asymptotes has constant area ab; and the parallelogram of tangents of conjugate diameters has constant area $4ab$;'

(ii.) 'The difference of squares of two conjugate semi-diameters has constant area $a^2 - b^2$;' ($CQ^2 - CR^2 = a^2 - b^2$);

(iii.) 'The rectangle of focal rays of a point is equal to the square of its conjugate semi-diameter.' ($FQ \cdot F'Q = CR^2$.)



(i.) If CQ, CR are conj. diams., and TQT′, TR tangts. to hyp. and conj., QE ∥ CT, 2α ang. of asms. CT, CT′; then RQ, CT are bisected at D, and ▱ RQ = △ TCT′ = $4CD \cdot CE \sin 2\alpha$.

If KQK′ ∥ CB, then $QK \cdot QK' = -b^2$, and trs. QDK, K′EQ are of const. form;

$\therefore$ $QD \cdot QE : QK \cdot QK' =$ const., μ say;

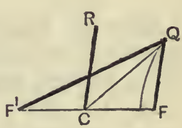
i.e. $CD \cdot CE = QD \cdot QE = -\mu b^2 =$ const.;

$\therefore$ △ TCT′ = ▱ RQ = $4CD \cdot CE \sin 2\alpha =$ const.

= ▱ BA = ab ($\because$ a, b are conj.).*

(ii.) If circ. on diam. TT′ cuts CT in H,

$$CQ^2 - CR^2 = CQ^2 - QT^2 = CH \cdot CT = CT' \cdot CT \cos 2\alpha$$
$$= \text{const.} = a^2 - b^2 \ (\because a,\ b \text{ are conj.}).$$



(iii.) $(2a)^2 = (FQ - F'Q)^2$

$= FQ^2 + F'Q^2 - 2FQ \cdot F'Q$

$= 2 \cdot CF^2 + 2CQ^2 - 2FQ \cdot F'Q$;

$\therefore$ $2a^2 = (a^2 + b^2) + (CR^2 + a^2 - b^2) - FQ \cdot F'Q$.

$\therefore$ $FQ \cdot F'Q = CR^2$.

Definition 11.—A rectangular or equilateral hyperbola has equal axes.

Theorem 30.—'In a rectangular hyperbola,'

(i.) 'The hyperbola ≡ its conjugate, from which it can be derived by rotation through a right angle.'

(ii.) 'The asymptotes are perpendicular.'

(iii.) 'Perpendicular diameters are equal.'

(iv.) 'Conjugate diameters are equal.' ($\because$ $a^2 - b^2 = 0$.)

* The movable tangt. TT′ forms homographic ranges on fixed tangts. CT, CT′, C in one corresp. to I at inf. in the other. Superposing these, we have an involn., centre C; $\therefore$ $CT \cdot CT' =$ const. (Pure geom.)

Theorem 31.—Apollonius' theorem: 'The ratio of the rectangles of parts of two transversals of given directions from any point to an ellipse or hyperbola is constant, and equal to that of the squares of semi-diameters of those directions.'*

If semi-diams. CR, CS ∥ transvls. PVP′, QVQ′ of given dirns., make HK polar of V cutting CV, RC at T, K, and make CM conj. to and bisecting PP′, and VL ∥ CM;
then polar of K on CR ∥ conj. CM, and traverses V, i.e. VL is polar of K;
∴ K, L are a pair of s.p. involn. of CR,
∴ $CR^2 = -CK \,.\, CL\dagger = CK \,.\, VM$.

Also, $VP \,.\, VP' = VH \,.\, VM$

(∵ HPVP′ is harm.),

$$\therefore \frac{VP \,.\, VP'}{CR^2} = \frac{VH}{CK} = \frac{VT}{CT} = \frac{VQ \,.\, VQ'}{CS^2} \text{ (simly.)};$$

i.e. $VP \,.\, VP' : VQ \,.\, VQ' = CR^2 : CS^2 = \text{const.}$

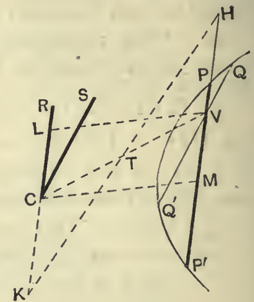
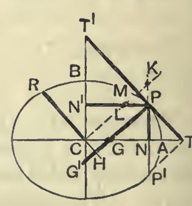


Fig. of Hyp.
(Same proof for Ell.)

Note. The case where QQ′ is the diameter conjugate to PP′ should be specially studied.

Theorem 32.—'If the ordinates, tangent, and normal of a point P of an ellipse or hyperbola meet the axes in N, N′, T, T′, G, G′, and PG meets CR, conjugate to CP, in H; then
(i.) $CN \,.\, CT = a^2$, $CN' \,.\, CT' = \pm b^2$;
(ii.) $PH \,.\, PG = \pm b^2$, $PH \,.\, PG' = a^2$.'

(i.) PNP′, chd. of contact of tangts. from T, is the polar of T (see Thh. 34, 36);
∴ $CN \,.\, CT = CA^2 = a^2$. And simly.
$CN' \,.\, CT' = CB^2 \text{ (ell.)} = C\beta^2 \text{ (hyp.)} = \pm b^2$.
(ii.) Draw CLMK perp. to TT′, parl. to PG;
∴ T′N′LM, TNMK are cyclic quadls.
∴ $PH \,.\, PG = CM \,.\, CL = CN' \,.\, CT' = \pm b^2$,
$PH \,.\, PG' = CM \,.\, CK = CN \,.\, CT = a^2$.



Students may learn Th. 36 here, corresp. to Thh. 6, 7, 8, Ch. VII.

* This statement should be learnt; the demonstration may be postponed.
† If R is a real point on the curve, the sign is positive.

EXAMPLES—XLVI.

CENTRAL CONICS.

1. Find the locus of the centre of a circle which touches two given circles. Discuss the different cases.

2. Construct an ellipse by pencil, loop of thread, and two pins.

3. Construct part of a hyperbola by pencil, thread, a rod movable about a fixed point in it, and a pin.

4. Focal radii of an ellipse or hyperbola increase with their angle from **FA**. (Use branch of hyp. in which **F** lies.)

5. Focal chords of a central conic increase with the angle from the latus rectum of their greater part.

6. The foci divide harmonically the axial intercept of tangent and normal of a point on a central conic.

7. If the normal and tangent of a point **P** of a central conic meet the axis in **G, T**, then $\mathsf{CG}\,.\,\mathsf{CT}=\mathsf{CF}^2$. (Use Ex. 6.)

8. Find the locus of a point whose distance from a given circle is proportional to its distance from a given straight line.

9. If **FM, FN** are perpendicular to the tangent and normal of a point on a conic, **MN** passes through the centre.

10. If the normal **PG** of a point to a conic is equal to **FG**, then **FP** is equal to the latus rectum.

11. Given the foci **F, F′** in position, and the axis $2a$ of a conic, construct the points **P, P′** of any focal ray.

12. A focal radius **FP** of a conic is produced to **Q**; show that **F′Q** is divided by the tangent of **P** in the ratio **PF′ : PQ**.

13. Any point on the central perpendicular to a tangent of a conic is equidistant from the feet of the focal perpendiculars.

14. Find the greatest intercept of a tangent to a conic by its axcircle.

15. Tangents at the ends of the two latera recta of a conic intersect on the axcircle or at infinity.

16. If the normal and tangent of a point on a conic meet **CB** in **G′, T′**, then $\mathsf{CG}'\,.\,\mathsf{CT}'=-\mathsf{CF}^2$.

17. A circle on a focal radius of a conic as diameter touches the axcircle.

18. The two circles through the point of contact of a tangent, one focus, and the foot of its perpendicular on the tangent, meet on the axis and on the ordinate of the point of contact.

19. Given the foci **F, F′** and the line of a tangent to a conic, construct the point of contact of the conic. What different cases are there?

20. Find the envelope of one side of a right angle whose vertex moves on a fixed circle and whose other side traverses a fixed point. (Ch. V.)

21. An arc of constant angle is described on a focal radius of a conic; show that it touches a fixed circle. (See Ex. 20 and Ch. V.)

22. The intersections of joins of ends of two parallel chords of a conic lie on the conjugate diameter.

23. The rectangle $QP \cdot QP'$ of the intercept by the axcircle on the tangent at Q of a conic is equal to $CR^2 - b^2$, where CR is conjugate to CQ.

24. If FM, $F'M'$ are perpendiculars from the foci to the tangent of a point P of a conic, and PN is an ordinate, then $NM : NM' = PM : PM'$.

25. Supplemental chords of a hyperbola are parallel to conjugate diameters.

26. A parabola is described through two foci of a hyperbola, with its focus on the curve. Show that its axis is parallel to an asymptote.

27. The asymptotes and a pair of conjugate diameters of a hyperbola form a harmonic pencil.

28. If PM is an ordinate to a diameter QQ' of an ellipse or hyperbola, RR' the conjugate diameter, $MP^2 : MQ \cdot MQ' = \pm CR^2 : CQ^2$.

29. If a diameter CQ meets an ordinate PM of a point P on a conic in M, and the tangent at P in T, show that $CM \cdot CT = CQ^2$. (Polar.)

30. A rod has two fixed studs on it, which move in two grooves forming a right angle. Show that a pencil fixed at a point in the rod traces an ellipse. (Trammels.) What is the curve if the grooves form any angle?

31. An ellipse and hyperbola have the same foci. Show that they cut at right angles.

32. If a parallelogram is inscribed in an ellipse, the centre is its diagonal point.

33. If a circle cuts a central conic in four points, any two common chords make equal angles with the axis.

34. Given in position two conjugate diameters of an ellipse, construct the curve. (If QQ', RR' are diams., constr. tangt. QT to meet director circ. in T; $TQ = b$, whence a.)

35. Given in position two conjugate diameters of a hyperbola, construct the curve. (Constr. asymptotes and axes.)

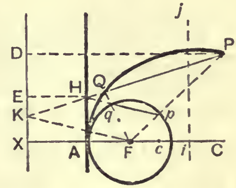
36. A chord of a conic which subtends a right angle at each focus is parallel to the axis.

37. A rectangular hyperbola circumscribing a triangle passes through its orthocentre.

38. The hyperbola approaches its asymptotes as it extends farther and farther from its vertex, and reaches them at infinity.

39. If two cars run, one along a hyperbola (sufficiently large), and the other along the near asymptote, starting from tangent of vertex and keeping their heads always on the same ordinate, show that they must collide.

Theorem 33.—'A conic is the perspective of its focircle, with focus and tangent of vertex as centre and axis of perspective.'



If a chd. PQ of a conic cuts drx. and tangt. of A in K, H, and PD, HE $\perp$ DX; draw Hqp parl. to FK to meet FQ, FP.

$\therefore$ Fp : FP = KH : KP = HE : PD = AX : PD,

$\therefore$ Fp : AX = FP : PD = e, i.e. Fp = FA;

$\therefore$ p, and simly. q, is on focircle. Hence:

Corresp. points P, q are collr. with F,

" lines PQ, pq " conc. on AH;

i.e. circle and conic are in persp., with F, AH as centre and axis.

Note. Small and large letters p, P, &c. on circ. and conic correspond.

Line at Infinity—Centre—Asymptotes.

Make Fi = XA, $\therefore$ Hi $\parallel$ FE; $\therefore$ ray EH is persp. of Hi;
$\therefore$ persp. of i is on EH, Fi, and is pt. at inf. I on XF; and the persp. of ij parl. to axis of persp. is IJ at inf. (Ch. VII., Th. 20.)

Also, if c is the pole of ij, its persp. C is the pole of IJ, and is the centre of the conic.

There are three cases, according to position of c and i.

(i.) i outside, c inside circle; FA < iF < AX; $e < 1$. (Above fig.)

The conic is an **ellipse**; ij does not meet the circle,

$\therefore$ IJ does not meet the ellipse; i.e.—

'An ellipse has no real points at infinity, and no real asymptotes.'

(ii.) i, c coincide on circle; FA = iF = AX; $e = 1$.

The conic is a **parabola**; ij touches the circle at c.

$\therefore$ IJ touches the parabola at inf. at its centre; i.e.—

'A parabola touches the line at infinity, which coincides with its asymptotes.'

(iii.) i inside, c outside circle; FA > iF > AX; $e > 1$.

The conic is a **hyperbola**; ij meets the circ. in real points j, j', from which real tangents cj, cj' are drawn from c;

$\therefore$ IJ meets hyp. in real points J, J'; CJ, CJ' are real tangts.; i.e.—

'A hyperbola meets the line at infinity in two real points, and has two real asymptotes.'

Ex. Construct, by perspective, the asymptotes of a hyperbola, FX = 1", $e = 3$. Compare with those of Constr. 5.

Theorem 34.—'In any conic or perspective of a circle,'

(i.) 'The polar of a point P in its plane is a straight line through the points of contact of tangents from P.'

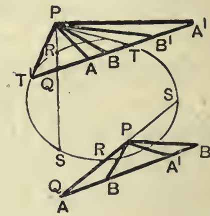
(ii.) 'If Q is on the polar of P, P is on the polar of Q.'

(iii.) 'The pencil of sides of self-polar triangles at a common vertex is in involution.'

'A line in its plane is cut in involution by the sides of self-polar triangles of its vertex.'

(iv.) 'The double $\frac{\text{points}}{\text{rays}}$ of the self-polar involution of a $\frac{\text{line}}{\text{point}}$ are the $\frac{\text{points of the ray on}}{\text{tangents from the point to}}$ the curve.'

(Ch. VII., Thh. 8, 9, 10, 27.)

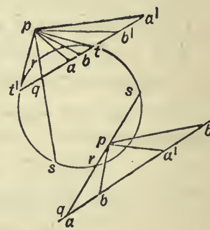


If PRS is a transvl. of P, Q the harm. conj. of P to R, S; then the polar of P is the locus of Q.

Project into a circle, denoting corresp. points by corresp. small letters; then,

(i.) $(prqs) = (\mathsf{PRQS}) = -1$,

∴ locus of q is a str. line ab, the polar of p to circle, through points of contact of tangts. pt, pt';



∴ locus of Q, i.e. the polar of P, is a str. line AB through points of contact of tangts. PT, PT′.

(ii.) If Q is on the polar of P, q is on polar of p, p on polar of q,

∴ P is on polar of Q.

(iii.) The self-polar pencil of P has P(AA′, BB′...) $= p(aa', bb'...)$;

∴ P(AA′, BB′...) and range (AA′, BB′...) are in involn.

(iv.) ab meets circ. in double points of $(aa', bb'...)$, and tangts. pt, pt' are double rays of $p(aa', bb'...)$;

∴ AB meets conic in double points of (AA′, BB′...), and tangts. PT, PT′ are double rays of P(AA′, BB′...).

Caution. Double points of an involn. project into double points, but the **centre does not project** in general into the centre. Explain this.

Ex.* Prove in the same way the cross-ratio properties of a conic. (Ch. VII., 21–24.)

* Important.

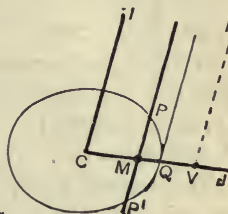
Theorem 35.—'In any conic or perspective of a circle,'

(i.) 'Pairs of conjugate diameters are pairs in the self-polar involution of the centre.'

(ii.) 'The polar of a point is conjugate to the central ray of that point, and parallel to the tangents of that ray.'

(iii.) 'The foot of the polar is the centre of the self-polar involution of that line.'

(A) Central curves. Centre at finite distance.

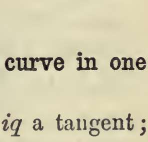


(i.) If CI, CJ (I, J at inf.) are a pair in the s.p. involn. of the centre,
CIJ is a s.p. triangle.
∴ transvl. IPMP′, parl. to CI, is divided harmy.;
∴ M bisects PP′, and CJ, CI are conj. diams.

(ii.) If PP′, polar of V, || CI,
then V is on polar of I, ∵ I is on polar of V;
∴ V is on CJ conjugate to CI or PP′.

(iii.) Draw VI parl. to PP′,
∴ IMV is a s.p. triangle;
∴ M, conj. of I in s.p. involn. of PP′, is centre of the involn., and also the foot of the polar of V.*

(B) Parabolic curves. Centre C at infinity.



C, on its own polar (line at inf.), is point of contact of tangt. polar; hence corresp. point *c* of circle is point of contact of corresp. tangt. *ci*. (Cp. Th. 33, ii.)

And every line through *c* meets the circle in one other real point *q*; hence:

'Each diameter of a parabolic curve meets the curve in one finite point; all diameters are parallel.'

Also, if *ipmp′* is polar of *v*, *cv* is polar of *i*, and *iq* a tangent; also (*cmqv*), (*ipmp′*) are harmonic.

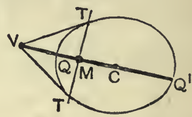
∴ PP′, polar of V, || tangt. QI (top figure, C, I at ∞);
M bisects PP′, and is centre of its s.p. involn.; Q bisects MV.

Hence (ii.), (iii.) above are true of parabolic curves; also:

'The diametral distance of a point from the foot of its polar to a parabolic curve is bisected by the curve.'

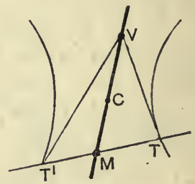
* The foot of polar to a conic is on central ray.

Theorem 36.—'If the central ray of a point V to a conic or perspective of a circle meets the curve in Q, Q′, and its polar or chord of contact of its tangents in M, then CM . CV = CQ².'



If TT′ is polar or chd. of contact of tangts. of a point V to the curve, then (VQMQ′) is a harm. range, and C is the mid point of QQ′;

$\therefore$ CM . CV = CQ².



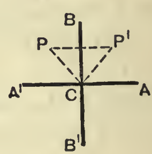
If VC meets the curve in imaginary points δ, δ_1, then δ, δ_1 are double points and C the centre of s.p. involn. of VC;

$\therefore$ CM . CV = Cδ^2. (Ch. VII., Thh. 26, 27.)

The corresponding property of a parabolic curve is VQ = QM.

Ex.* 'The focus is the pole of the directrix, and its s.p. involution is right-angled.'

Theorem 37.—'One pair of conjugate diameters of a conic or perspective of a circle is right-angled, and each of these is a line of symmetry.'



The conj. diams. form the s.p. involn. of the centre, which has one rt.-angled pair AA′, BB′. (Ch. VII., Th. 30.)

$\therefore$ chd. PP′, parl. to AA′, is bisected by its conj. CB;

$\therefore$ curve is symmetrical about BB′, and simly. about AA′.

Cor.—'If the central involution of a conic is right-angled, the conic is a circle.'

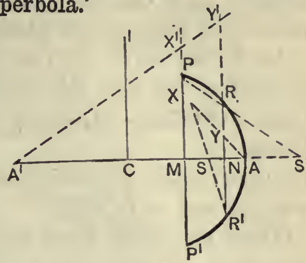
For the curve is then symmetrical about every diameter. Hence, if a diameter BB′ bisects the angle PCP′ of any two central radii, CP = CP′.

Note. If the s.p. involn. of a point is right-angled, it is easy to show that the point is a focus, and its polar a directrix. (Ex. 7, below.)

These points must come on the principal axes (the right-angled pair of the centre), and two of them are always imaginary (on the minor or transverse axis). Tangents from a focus pass through the circular points at infinity (Ch. VII., Th. 30).

* Important.

Theorem 38.—'Every perspective of a circle or conic, centre at finite distance, is an ellipse or hyperbola.'



One at least of the right-angled pair of conj. diams. meets the curve in real points A, A'.*

If CI (I at inf.) is conj. to AA', and chds. PP', RR' || CI; complete quadl. PP'R'R, to I, S'; draw diags. PSR', P'SR.

∴ ISS' is a s.p. triangle (Ch. VII., Constr. 2); ∴ S, S' are on AA' and are harm. conjs. of A, A'.

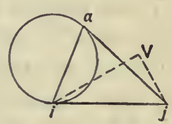
Take A, X, Y collr. on diags. SS', PP', RR' of quadl. SRS'R';

∴ their harm. conjs. A', X', Y' are collinear (Ch. VII., Th. 22); also $MP^2 = MX \cdot MX'$, $NR^2 = NY \cdot NY'$ (Ch. VII., Th. 5);

$$\therefore \frac{MP^2}{MA \cdot MA'} = \frac{MX \cdot MX'}{MA \cdot MA'} = \frac{NY \cdot NY'}{NA \cdot NA'} \text{ (sim. trs.)} = \frac{NR^2}{NA \cdot NA'} = \text{const.};$$

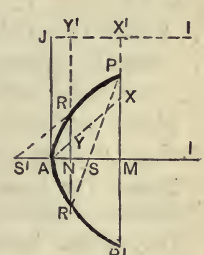
∴ locus of P, persp. of circle, is an ellipse or hyperbola (Th. 18).

Theorem 39.—'Every perspective of a circle or conic, centre at infinity, is a parabola.'



If ji, tangt. at i to circ., projects to inf. (V centre of persp.), and $Vj \perp Vi$, and ja is the other tangt. of j; then aj, ai proj. into AJ, AI parl. to Vj, Vi;

∴ JAI is a rt. ang. (Ch. VII., Constr. 6, ii.).



If chds. RR', PP' || AJ; complete quadl. PRR'P' and diags. to J, S', S;

∴ SS' along AI is divided harmy. at A, I.

Make MX, NY eql. to AM, AN;

∴ A, X, Y are collr. on diags. SS', PP', RR' of quadl. SRS'R';

∴ their harm. conjs. I, X', Y' are collinear;

∴ X'Y' || AI, and $MX' = NY' = \text{const.} = l$ say;

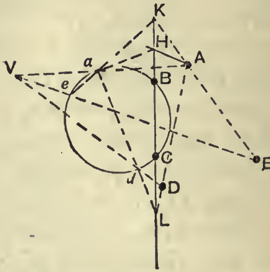
∴ $MP^2 = MX \cdot MX' = l \cdot MX = l \cdot AM$.

∴ locus of P, persp. of circle, is a parabola whose vertex is A, and whose focal distance AF is $l/4$.

* Because two of three sides of a s.p. tr. of a circle meet it in real points.

Theorem 40.—'Any five points can be projected on to a circle;* and one only conic† can be drawn through any five points (no three points being collinear).'

If A, B, C, D, E are five points, determine the line AH through A so that A(HBCD) = E(ABCD) (Ch. VII., Constr. 4);



then AH cuts externally one at least side, BC say, of tr. EBC.

Draw any circle BC*a*, H*a* a tangent; join EA to K; K*a* to *e*; A*a*, E*e* to V;

∴ HBC*ea* is persp. of HBCEA to vertex V, axis BC.

Join AD to L, L*a* to meet circ. in *d*.

∴ *e*(*a*BC*d*) = *a*(*a*BC*d*) = (HBCL), (*aa* is tangt. at *a*),
= A(HBCD) = E(ABCD);

and three pairs of corresponding rays meet in K, B, C on BC;

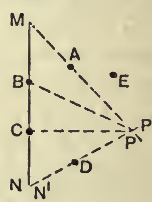
∴ the fourth pair *ed*, ED must meet on BC;

∴ triangles *ead*, EAD are in persp. (Ch. VII., Th. 17, ii.);

∴ D*d*, E*e*, A*a* are concurrent in V;

∴ *a*BC*de* is perspective of ABCDE.

Hence a conic, the persp. of circle *aed*, passes through the five points A, B, C, D, E.



Also if a line AP meets this conic in P, and any conic whatever through the five points in P′; join PA to M; PD, P′D to N, N′ on BC.

∴ P′(ABCD) = E(ABCD) = P(ABCD);

∴ (MBCN) = (MBCN′);

∴ N coincides with N′, and P′ with P.

Hence any conic through the five points coincides with this conic—i.e. there is one only conic.

Note. The conic is the locus of points P whose cross ratio P(ABCD) = E(ABCD). Hence:

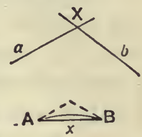
Cor.—'The locus of points whose pencil to four fixed points is constant, is a conic.'

(See Ch. VII., Constr. 4, Th. 16. AH (top fig.) is tangt. at A.)

* The construction given is always real.

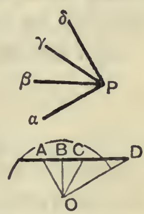
† A conic is assumed to have the cross-ratio properties of a circle. These follow exactly as in Th. 34.

If all points and lines of a figure are replaced respectively by their polars and poles to a fixed circle or conic, the resulting figure is the **polar reciprocal,** and the process is called **reciprocation.**



The polar reciprocal of a curve AB is the envelope of polars a, b of points A, B on it. Their intersection X is the pole of the chord AB or x; and as B moves to coincidence with A, b moves to coincidence with a; x becomes the tangent at A, and X the intersection of coincident tangents—i.e. the point of contact of a—on the envelope (Ch. V., Envelopes).

Theorem 41.—'The polar reciprocal of any range is a homographic pencil.'



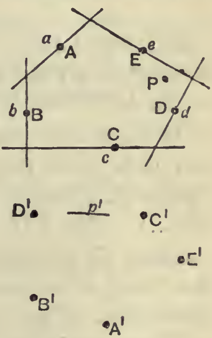
If ABCD is the range, P its pole to a circle, centre O, P$(\alpha\beta\gamma\delta)$ the polar reciprocal of (ABCD); then P$(\alpha\beta\gamma\delta) \perp$ O(ABCD), in the circle,

$\therefore$ P$(\alpha\beta\gamma\delta) \equiv$ O(ABCD) $=$ (ABCD).

And by projecting a conic into a circle, the theorem is seen to be true for any conic.

Theorem 42.—(i.) 'The polar reciprocal of a conic is a conic.'

(ii.) 'One only conic can be drawn to touch five straight lines' (no three lines being concurrent).



If a, b, c, d, e are the lines; A′, B′, C′, D′, E′ their poles to a given conic; draw tangts. a', b', c', d', e' to the conic A′B′C′D′E′ at these points.

The polar recpl. of this conic is a curve touching a, b, c, d, e at A, B, C, D, E, say; $\therefore$ if P is any point on this curve, its polar recpl. p' touches conic A′B′...E′.

$\therefore$ P(ABCD) $= p'(a'b'c'd') = e'(a'b'c'd')$ (Ch. VII., Th. 25)

$=$ E(ABCD).

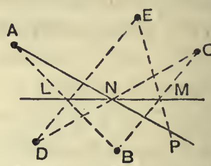
$\therefore$ locus of P is a conic touching a, b, c, d, e at A, B, C, D, E.

Hence the polar reciprocal of a conic A′B′C′D′E′ is a conic, and any conic touching a, b, c, d, e must reciprocate into A′B′C′D′E′.

$\therefore$ there is only one conic touching these lines.

Cor.—'The envelope of a line forming a range of constant cross ratio on four given lines is a conic.'

Construction 6.—'Construct a conic through five points.'



We have already given two solutions of this problem. In Ch. VII., Constr. 4, the point P on a transversal AP of the conic is determined by the cross-ratio property $P(ABCD)=\mu$, μ being here equal to $E(ABCD)$.

And in Th. 40 we show how the conic may be derived from a circle by perspective. But Pascal's theorem gives a simple solution by drawing straight lines only, without any calculation. Thus, given five points A, B, C, D, E; then if P is the point of the conic on a ray AP, ABCDEP is an in-hexagon.

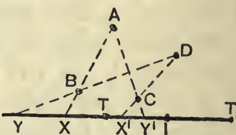
Join AB, DE to L; AP, DC to N; BC, LN to M; EM to P.

CONICS THROUGH FOUR POINTS.

Since a conic can be drawn through five points, any number can be drawn through four points. But a parabola has already one condition given—viz. the tangent line at infinity; hence only a limited number of parabolas can be drawn through four points.

Construction 7.—(i.) 'Construct a conic through four points to touch a given line;

(ii.) Construct the parabolas through four points.'



The given tangent TT_1 cuts the conic and opp. sides AB, CD and BD, AC of the quadl. of the points in the involn. $(XX', YY'\ldots)$. (Ch. VII., Th. 33.)

Since TT_1 touches the required conic, the point of contact is one of the double points T, T_1 of this involution. Thus five points are known.

There are clearly two solutions.

For the parabola, construct the double rays BT, BT_1 of the pencil $B(XX', YY'\ldots)$ on any transversal; then BT or BT_1 passes through the point of contact of the parabola at infinity. Thus a fifth point on the parabola is known—viz. the point at inf. on a double ray—and two only parabolas can be drawn through four points. Similarly, one parabola can be drawn to touch four lines.

EXAMPLES—XLVII.

1.* The curve $MP^2 : MQ . MQ' = \mu$, a constant, MP of given direction, M on a fixed length QQ', is a conic. Interpret for parabola.

2. If a quadrilateral ABCD is inscribed in a conic, the intersections of AB, CD; AD, BC; and AC, BD form a self-polar triangle.

3. Project a circle or conic into another conic so that any given point in its plane becomes the centre of the new curve.

If the given point is on the curve, what is the conic?

4. A conic P is projected into another P′ so that a given line projects to infinity. What do the sides of s.p. triangles of the pole of this line become? What is the special property of these lines?

5. Any two angles can be projected into right angles, and at the same time any straight line to infinity (Ch. VII., Constr. 6). Apply this to Ex. 4 so that P′ may be a circle. Is the construction always real? (See Th. 37, Cor.)

6. If any point is inside a conic (so that real tangents cannot be drawn from it), every straight line through it cuts the curve.

7. If the s.p. involution of a point F to a conic is right-angled, and DX is polar of F, if P is any point on the curve, and if FX perpendicular to DX meets the curve in A; show that

(i.) Every line through F meets the conic.

(ii.) A tangent PD from D on the directrix subtends a right angle at F.

(iii.) If PA meets DX in K, then K is a bisector of AFP (polar of $K \perp KF$ and divides KAP harmonically).

(iv.) F is a focus (show $FP : PM = FA : AX$, if $PM \perp DX$).

8. If the distances PF, QF, RF of three points on a conic from F are proportional to their distances PL, QM, RN from the polar of F, show that F is a focus.

9. Assuming that the s.p. involution of a focus of a conic is right-angled, show that the focus must lie on one of the right-angled pair of conjugate diameters.

10. Show that real tangents can always be constructed from two only of the vertices of a s.p. triangle of a conic.

11. Real points exist on two only of the sides of a s.p. triangle. How do these sides correspond to the vertices of Ex. 10? Why?

12. Given the centre and a s.p. triangle of a conic, construct the real points on the sides of the triangle. Simplify this when the triangle is obtuse and the point is the orthocentre.

13. Show that there is one only conic having a given centre and a given s.p. triangle. If the triangle is acute and the point its orthocentre, what is the conic? (Show that five points are fixed.)

* Important.

14. Right-angled pairs of chords of a conic from a point on it determine a system of chords concurrent on the normal. (Ch. VII., Th. 32.)

15. The circumcircles of s.p. triangles of a point to a conic are coaxial.

16. The intercept of a tangent of a conic by the directrices is divided harmonically by the point of contact and the pole of the normal.

Interpret this for the parabola.

17. Find the locus of the pole of the side AD to all conics circumscribing the quadrilateral ABCD.

18. Show that a conic can be circumscribed about a quadrilateral ABCD so as to have any side, or the diagonal AC or BD, a diameter.

19. If three conics have a common chord, the other common chords of each two of them are concurrent.

20. ABCD is a quadrilateral, AB is fixed, and CD of given length moves on a straight line. Find the locus of intersections of AD and BC.

21. The locus of the point P on a transversal QXYZ of a fixed triangle ABC, turning about a fixed point Q, such that $(XYZP)=\mu$, a constant, is a conic through A, B, C, Q.

22. If three sides of a variable triangle pass through fixed points, and two vertices move on fixed lines, the locus of the third vertex is a conic. (MacLaurin.)

23. If three vertices of a triangle move on fixed lines, and two sides pass through fixed points, the third side envelops a conic. (Reciprocate the last example.)

24. Two tangents from a point to a conic form an involution with the joins of the point to the vertices of a circumscribing quadrilateral of a conic. Reciprocate this.

25. Reciprocate Pascal's theorem and Brianchon's theorem.

26. Reciprocate with regard to a circle the theorem, 'The locus of a point at which two fixed points subtend a right angle is a circle.'

And the following: 'The envelope of one side of a right angle whose vertex describes a fixed line, and whose second side traverses a fixed point, is a conic.'

27. If the polars to a conic of three vertices of a triangle are concurrent, the polars of all points are concurrent. What is the conic?

28. Two angles of constant magnitude move about fixed points P, Q, and the intersection of one side of each describes a straight line; show that the intersection of the other sides describes a conic. (Newton.)

What is the curve if the first intersection describes a conic?

29. Find the locus of the centres of all conics passing through four given points. Where does it meet the chords of these points?

30. Four points A, B, C, D of a variable conic are fixed. Two fixed lines AP, CQ meet the conic in P, Q. Show that PQ passes through a fixed point.

CHAPTER IX.

SOLID GEOMETRY.

PLANES, POLYHEDRA, CONE, CYLINDER, SPHERE.

Definition 1.—The **meet** of two planes which have a common point is the straight line in which they cut each other.

Definition 2.—A **dihedral angle** is the figure of two planes which terminate in a common line (e.g. angle of two pages of a partly opened book).

Definition 3.—A **plane section** of a figure is a figure formed by the first figure on a plane cutting it.

Definition 4.—A **perpendicular to a plane** is a straight line perpendicular to all lines in the plane which meet it.

Definition 5.—A **straight line parallel to a plane** is a parallel to some line in the plane.

Definition 6.—**Parallel planes** are perpendicular to the same straight line.

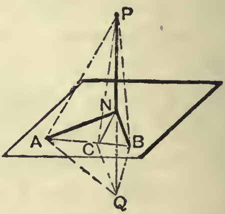
Note. It is shown below that one only perp. to a plane exists at each point in the plane, and that any two such perps. are parl. The direction of a perp. therefore serves to define the direction of the plane. Thus two parallel planes as just defined have the same direction, which is consistent with our definition of parallel straight lines. It is shown also that parl. planes either coincide or do not meet, and that non-parl. planes meet.

Conditions which determine a Plane.

Either part of a plane may be turned about the line joining two fixed points A, B in it; if the two parts are turned at a suitable rate, they may be kept always in one plane, and in making a complete or half turn they sweep out the whole of what is called space—i.e. in some position or other they contain any third point C whatever; thus a plane can be drawn through any three non-collinear points A, B, C by moving to pass through A, B and turning to pass through C; and any second plane through A, B, C coincides with the first (Th. 5, Ch. I.). Hence:

Theorem 1.—'**One only plane can be drawn through any three non-collinear points; through any straight line and a point not on the line; through any two intersecting or parallel straight lines.**'

Theorem 2.—'**A straight line perpendicular to two straight lines in a plane is perpendicular to all straight lines in the plane through their common point; i.e. it is perpendicular to the plane.**'



If NP $\perp$ NA, NB in plane ANB, and NC is any other line in the plane, produce PN to Q, make NQ eql. to NP.

Then in plane APQ, NA is rt. bisector of PQ,

$\therefore$ AP = AQ; similarly BP = BQ; and AB is common to trs. APB, AQB,

$\therefore$ ang. PAB = QAB.

And in trs. APC, AQC, AQ = AP, AC is common, ang. PAC = QAC;

$\therefore$ CP = CQ, and C is on rt. bisector of PQ in plane PCQ;

$\therefore$ CN $\perp$ PQ.

Hence PN $\perp$ any other line NC in plane ANB.

Cor. (i.).—'**The locus of perpendiculars to a line at a given point is the plane perpendicular to the line at that point.**'

Cor. (ii.).—'**The locus of points in space equidistant from two fixed points is the plane bisecting their join at right angles.**'

Construction 1.—'**Construct a plane perpendicular to a given line at a given point.**'

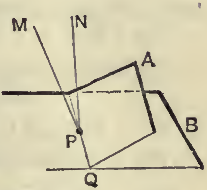
Draw a perp. to the line at the point, and rotate the figure about the line; the perp. sweeps out the plane.

Construction 2.—'**Construct a perpendicular to a plane at a point in it.**'

Construct any plane P by (i.) perp. to a line X; superpose P on the given plane so that X passes through the given point; the new position of X is the required perpendicular. Clearly there is one only perp. at each point of a plane.

Theorem 3.—'Two planes having a common point either coincide or meet in a straight line.'

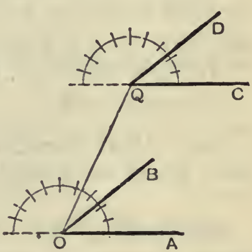
If planes A, B meet at P, draw the perps. PM, PN to A, B; then
either (i.) PM coincides with PN,
∴ plane A coincides with B;
or (ii.) PM does not coincide with PN;
draw PQ perp. to plane NPM,
∴ PQ ⊥ PM and PN; i.e. PQ is in planes A, B;
∴ planes A, B meet in a straight line PQ.



Theorem 4.—'The angle formed by two planes on a third plane perpendicular to their meet is constant, and called the angle of the planes.'

If AOB, CQD are angles formed by planes AQ, BQ on planes AB, CD perp. to their meet OQ:

In plane AOB make a scale, unit two right angles, from OA, and divide decimally the unit, tenth, &c. containing OB.



Planes through OQ and the points of division determine a similar scale, same unit, from QC in plane CQD;
and OB, QD come between the same divisions of the two scales;
∴ ang. AOB = CQD = constant.

Note. Perpendiculars to OA, OB in plane OAB ⊥ planes AQ, BQ, and their angle = BOA. Hence:

Cor. (i.).—'The angle of two planes is equal to that of their perpendiculars at any point on their meet.'

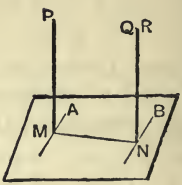
Cor. (ii.).—'A plane containing a perpendicular to a given plane is perpendicular to the plane.' *

Cor. (iii.).—'The meet of two planes each perpendicular to a third, is perpendicular to the third plane.'

Ex. Find the shortest distance from a given point to a given plane.

* ∵ the angle of the planes is right.

Theorem 5.—'Two perpendiculars to a plane are parallel.'



If MP, NQ ⊥ plane AB,
make MA, NB perp. to MN, and
make NR parl. to PM in plane PNM;
∴ NR ⊥ MN, and MN ⊥ plane RNB.
∴ ang. RNB = ang. of planes PNM, AB
= PMA = rt. ang.
∴ NR ⊥ NB and to NM, i.e. ⊥ plane AB;
∴ NQ coincides with NR and ∥ MP.

Cor.—'Perpendiculars to parallel planes are parallel.'

Theorem 6.—'A plane perpendicular to any line is perpendicular to any parallel line.'

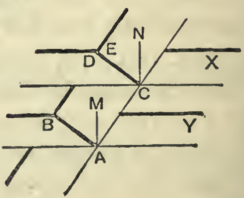
If plane AB (last fig.) ⊥ MP, and NR ∥ MP,
then NR is in plane PNM;
make NQ perp. to plane AB.
∴ NQ ∥ MP ∥ NR; i.e. NQ coincides with NR;
∴ plane AB ⊥ NR.

Cor.—'Two parallels to a third line (which need not be in their plane) are parallel.'

Construction 3.—'Construct a perpendicular to a given plane AB from an outside point Q.'

Make MP ⊥ AB, QN ∥ MP; ∴ QN ⊥ AB.

Theorem 7.—'The meets of parallel planes with any third plane are parallel.'



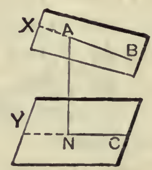
If parl. planes X, Y meet a plane AD in CD, AB;
make plane NAC perp. to AB, containing the parls. CN, AM perp. to X, Y;
and make CE parl. to AB in plane AD.
∴ CE ⊥ plane NAC, ⊥ NC;
∴ CE is in plane X, and also in plane AD;
∴ CE is the meet of planes X, AD;
∴ CD coincides with CE and ∥ AB.

Theorem 8.—'Two parallel planes either coincide or do not meet.'

Parallel planes X, Y
either (i.) have a common point N and a common perp. NP, and therefore coincide;
or (ii.) do not meet. (We say that they meet in the str. line at infinity.)

Theorem 9.—'Two non-parallel planes meet.'

If plane X not || Y, and from a point A in X,
AN is perp. to Y;
choose some line AB in X not perp. to AN.
∴ in plane BAN, cutting Y in NC,
AB not || NC, and meets it;
∴ the planes X, Y meet.

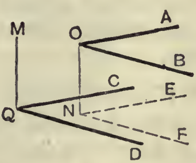


Theorem 10.—'Three planes, no two of which are parallel, meet in a point, which may be at infinity.'

If a plane Z (BAN in last fig.) meets two planes X, Y in AB, NC; then
either (i.) AB, NC in plane Z cross at a point,
or (ii.) AB, NC " Z are parallel and meet at infinity.

Theorem 11.—'Two lines which meet form the same angle as any two parallels to them which meet; and the planes of the two pairs of lines are parallel.'

If OA || QC, and OB || QD,
make ON perp. to plane AOB,
meeting plane CQD in N; *
draw NE, NF parl. to CQ, QD;
∴ NE || OA ⊥ ON; and NF || OB ⊥ ON;
∴ plane ENF, i.e. CQD ⊥ ON
|| plane AOB.



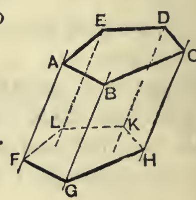
Also, ang. AOB = ang. of planes AON, BON = ENF
= CQD.

* If QM ⊥ plane CQD, the parl. ON meets plane CQD on the meet of the planes OMQ, CQD.

Theorem 12.—'The meets of planes parallel to a given line are parallel to this line.'

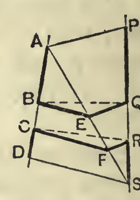
For a parl. to the line, through any point on the meet of two planes parallel to it, is in both planes and coincides with their meet.

Theorem 13.—'The sections by parallel planes of a system of planes parallel to a given line are congruent.'



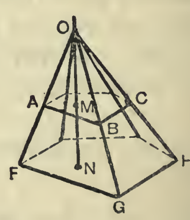
If a system of planes AG, BH, &c. parl. to a given line meet in AF, BG, &c., and cut two parl. planes in polygons ABC..., FGH...;
then AF || given line || BG, &c.;
AB || FG; BC || GH, &c.;
∴ ang. ABC = FGH, &c.; and figs. AG, BH, &c. are parms.;
∴ AB = FG, BC = GH, &c.;
∴ polygon ABC... ≡ poln. FGH....

Theorem 14.—'A system of parallel planes cuts off proportional parts from any two straight lines.'



If parl. planes cut two lines in A, B, C, D and P, Q, R, S, and AS in A, E, F, S;
then BE || CF || DS;
AP || EQ || FR;
∴ AB : CD = AE : FS = PQ : RS.

Theorem 15.—'The sections by parallel planes of a system of concurrent planes are similar polygons whose areas are proportional to the squares of their distances from the common vertex.'

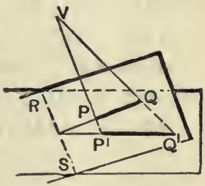


If planes AG, BH, &c. through O meet in OA, OB, &c., and cut two parl. planes in polns. ABC..., FGH...; and OMN ⊥ ABC;
then tr. OAB ||| OFG, and OMB ||| ONG;
∴ AB : FG = OB : OG = BC : GH, &c.;
ang. ABC = FGH, &c.;
∴ poln. ABC... ||| FGH....

Also, area AC : area FH = $AB^2 : FG^2 = OM^2 : ON^2$.

If all the points of a figure are joined to a fixed point, forming a pencil or cone of rays at the point, any plane section of the pencil is a representation of the figure, and is called its perspective or projection on the plane; the fixed point being the origin or vertex of projection. Thus the representation of a scene on a painter's canvas is the projection of the scene on the plane of the canvas from the eye as vertex. We need here only consider the projection of plane figures.

If V is the vertex of projection, P, Q points on a given plane A; P′, Q′ their projections on another plane A′, and RS the meet of A, A′;



then the plane VPQ contains P′Q′, and the meets of the planes A, A′, VPQ have a common point;
i.e. PQ, P′Q′ intersect on the meet RS of the planes A, A′.

Hence the important principle in the projections of plane figures:

Theorem 16.—'The joins of corresponding points of two plane figures in perspective pass through the vertex, and the intersections of corresponding straight lines are on the axis of projection.'

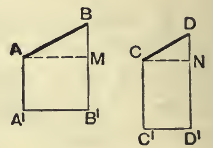
It is clear, therefore, that the relations of two such figures are precisely the same as in plane perspective.

Thus the projection of a circle is a conic, and the projection of a conic is a conic; also, any plane section of a cone on a circle or conic as base is a conic.

All points at infinity in one plane correspond to a straight line on the plane of projection, parallel to the meet of planes; so that, for example, a parallelogram projects into a complete quadrilateral. Conics were originally studied as plane sections of the right cone, whence the name; and the modern geometry of position, developed by Cremona and others, is based upon projection in space, of which plane projection is a limiting case; the conic being defined as the projection of a circle. Students who have mastered Chh. VII. and VIII. will have no difficulty in following the more general theory of perspective in space.

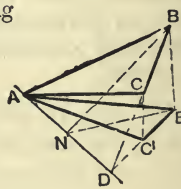
If perpendiculars from all points of a figure are drawn to a given plane, the plane figure formed is the right projection of the given figure on the plane. By Th. 13, projections on parallel planes are congruent.

Theorem 17.—'The right projections of parallel lines on a plane are parallel, and proportional to the lines.'



If $A'B'$, $C'D'$ are rt. projs. of AB, CD on a plane,
and AM, $CN \perp BB'$, DD';
then $BB' \parallel DD'$, and $AB \parallel CD$,
$\therefore$ plane $AB' \parallel$ plane CD', $\therefore A'B' \parallel C'D'$,
and ang. $B = D$, $\therefore$ rt. triangle $ABM \parallel\!| CDN$;
$\therefore AB : CD = AM : CN = A'B' : C'D'$.

Theorem 18.—'The area of the right projection of a plane area on a plane is the product of the given area and the cosine of the angle of the planes.'



If plane $AB'D \parallel$ plane of projection, meeting plane of tr. ABC in AD,
and plane $BNB' \perp$ meet AD;
ang. $BNB' =$ ang. of planes $= \alpha$ say.
$\therefore$ area $AB'D = \frac{1}{2}AD \,.\, B'N = \frac{1}{2}AD \,.\, BN \cos \alpha$
$= ABD \cos \alpha$.

Similarly, $AC'D = ACD \cos \alpha$,
$\therefore$ sum or diffce. $AB'C' = ABC \cos \alpha$.

Similarly, if any area X is broken up into triangles, proj. $X' = X \cos \alpha$.

Plan and Elevation.

Drawings of solid figures for machine and building construction are made by means of right projections on horizontal and vertical planes, called plan and elevation respectively. The methods of construction are given in text-books of Practical Solid Geometry. The projection of a circle, radius r, is an ellipse, axes $2r$, $2r \cos \alpha$. Also, the projection of a parallogram is a parallelogram.

Definition 7.—A **solid figure** is one whose points, lines, surfaces are not all in one plane.

Definition 8.—A **conical surface** is one generated by a straight line moving so as always to pass through a fixed point.

The moving line in any particular position is a **generator,** and the fixed point the **vertex** of the surface.

Definition 9.—A **cylindrical surface** is one generated by a straight line parallel to a fixed line, and moving along a given curve.

The fixed line is **the guide,** and the moving line in any particular position a **generator,** of the surface.

Definition 10.—A **solid angle** is a figure formed by three or more planes, each meeting two others, at a point (**polyhedral angle**), or by a conical surface which returns into itself.

Polyhedral angles are **trihedral, tetrahedral,** &c. according to the number 3, 4, &c. of planes at the point.

Definition 11.—A **polyhedron** is a figure enclosed by planes. A polygon formed with one of the planes by its neighbours is a **face,** a side of a face an **edge.**

A **regular polyhedron** has all its faces congruent regular polygons.

Polyhedra are **tetrahedron, hexahedron,** &c. according to the number 4, 6, &c. of faces.

A **paralhedron** * is a **hexahedron** whose faces are parallel, two and two.

Its faces are parallelograms and its edges are equal, four and four.

A **cuboid,** or **rectangular paralhedron,** is a paralhedron whose non-parallel faces are perpendicular.

Its faces are rectangles and its edges are equal, four and four.

A **cube** is an equal-faced cuboid.

Its faces are squares, and all its edges are equal.

* This name is more suggestive than parallelepiped.

Definition 12.—A **pyramid** is a figure formed by a polyhedral angle and a plane cutting its faces.

The point of the angle is **vertex,** the plane section **base,** and the distance vertex to base the **altitude,** of the pyramid.

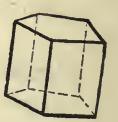
The faces meeting at the vertex are triangles.

Definition 13.—A **cone** is a figure formed by a conical angle and a plane cutting it.

The point of the angle is **vertex,** the plane section **base,** and the distance vertex to base the **altitude,** of the cone. A straight line along the surface from vertex to base is a **vertical edge.**

A **right circular cone,** or **cone of revolution,** has its base a circle, and its vertex on the axis or central perpendicular of the base. The length of a vertical edge is in this case the **slant height** of the cone.

Definition 14.—A **prism** is a polyhedron having two parallel faces, and the others parallel to a given line called its **guide.**



The guide faces are parallelograms, and the guide edges all equal; the parallel bases are congruent polygons, and their perpendicular distance is the **altitude** of the prism.

Prisms are **triangular, hexagonal,** &c. according to the number of sides 3, 6, &c. of the base.

A **wedge** is a triangular prism.

Definition 15.—A **cylinder** is a figure enclosed by a cylindrical surface and two parallel planes.

The plane sections are **bases,** their perp. distances the **altitude,** of the cylinder, and straight lines along the surface from base to base, parl. to the guide, are **guide edges.**

A **right cylinder** has its bases perpendicular to the guide.

A **right circular cylinder** is a right cylinder with circular bases. The central perpendicular of its bases is its **axis.**

Note. Theorems 13 and 15 may be extended to cylinders and cones by considering these as limiting forms of pyramid or prism whose vertical or guide edges move up to coincidence. Thus:

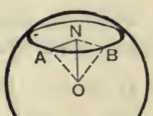
Theorem 19.—(i.) 'Parallel plane sections of a cylinder are congruent; (ii.) parallel plane sections of a cone are similar, and their areas are proportional to the squares of their distances from the vertex.'

Definition 16.—A **sphere** is a closed surface all points of which are equally distant from a fixed point, called its **centre.**

A straight line from centre to surface is a **radius,** and from surface to surface through the centre a **diameter.**

Note. A right circular cone, a right circular cylinder, and a sphere can be generated by revolving respectively a right triangle, a rectangle, and a semicircle about one side.

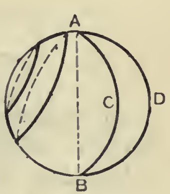
Theorem 20.—'A plane section of a sphere is a circle, and is perpendicular to the diameter through the centre of the circle.'



If AB is a plane section of a sphere, centre O,
and ON $\perp$ plane AB;
then in rt. trs. OAN, OBN,
OA = OB, ON is common,
$\therefore$ NB = NA.
$\therefore$ B is on the circle, centre N, rad. NA;
and the plane $\perp$ the diameter NO.

Definition 17.—A central plane section of a sphere is a **great circle**; any other plane section a **small circle.**

Definition 18.—A **lune** of a sphere is a portion of the surface cut out by a dihedral angle through a diameter (as ACBD).

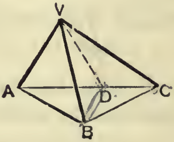


Definition 19.—A **zone** is a portion of a sphere cut out by two parallel planes.

Definition 20.—A **tangent plane** to a sphere at a point is the plane containing all tangents to great circles through the point; and it is perpendicular to the radius at that point.

By joining any number of points to form an inpolyhedron, and drawing tangents at these points to form a circumpolyhedron, the sphere can be seen to be the limit of an inpolyhedron whose vertices move up to coincidence, their number becoming infinite.

Theorem 21.—'The sum of any two face angles of a trihedral angle is greater than the third.'



If the face ang. AVC is not less than either ang. AVB, BVC of trihedral ang. V; make ang. AVD eql. to AVB, and make VB = VD.

$\therefore$ tr. AVB $\equiv$ AVD, and AB = AD.

But AB + BC > AC, in tr. ABC,

> AD + DC;

$\therefore$ BC > DC.

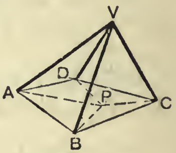
And in trs. BVC, DVC,

BV = DV, CV is common, BC > DC;

$\therefore$ ang. BVC > DVC.

$\therefore$ angs. AVB + BVC > AVD + DVC > AVC.

Theorem 22.—'The sum of face angles of a polyhedral angle is less than four right angles.' *



If ABC... is a plane section of a polyhedral angle V, and P a point within the polygon ABCD...;

then sum of angs. of all trs. at V

= sum of angs. of all trs. at P.

But sum of base angs. A, B, C... of the V trs.

> sum of base angs. A, B, C... of the P trs.;

$\therefore$ sum of face angs. at V < sum of angs. at P

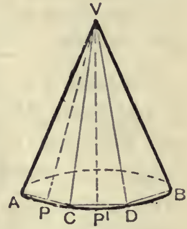
< four rt. angs.

Regular Polyhedra.

No regular polyhedron can have face polygons of more than five sides, since three angles of a regular hexagon make 4 rt. angs., so that three or more angles of a regular hexagon, heptagon, &c. cannot form a polyhedral angle; thus all the possibles are Tetrahedron, Octahedron, Icosahedron (3, 4, 5 triangles at a point); Cube (3 squares at a point); Dodecahedron (3 pentagons at a point). These may be studied from models.

* If there are re-entrant angles, the theorem may not be true.

Theorem 23.—'The area of the curved surface of a right circular cone is half the product of slant height and circumference of base.'



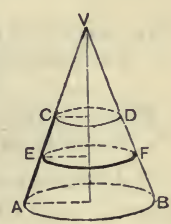
If **VAB** is a cone, **VA, VC, VD**... vertical edges forming the pyramid **VACD**..., and **VP** perp. to **AC**; then
the surface of the cone is the limit of the sum of tr. faces **VAC, VCD**..., when the edges of the pyramid move up to coincidence; and its base is the limit of the polygon **ACD**....

$$\therefore \text{ area of cone } \mathsf{VAB} = \text{limit of } \left(\frac{\mathsf{VP}\,.\,\mathsf{AC}}{2} + \frac{\mathsf{VP}'.\,\mathsf{CD}}{2} + \ldots\right)$$
$$= \mathsf{VA} \times \text{half length of circle } \mathsf{AB}$$
$$= \text{half (slant height} \times \text{circumf. of base).}$$

Theorem 24.—'The curved surface of a right cylinder is the product of altitude and circumference of base.'

Prove as the last, using guide edges.

Theorem 25.—'The area of the curved surface of a frustum * of a right circular cone is the product of its slant height and circumference of median circle.'



If **CDAB** is a frustum of a rt. circ. cone, vertex **V**, slant height h, base radii r', r_1; and if h', h_1 are slant heights of the cones **VCD, VAB**; then diam. of median circle

$$= \tfrac{1}{2}\,(\text{diam. } \mathsf{CD} + \text{diam. } \mathsf{AB})$$
$$= r' + r_1;$$

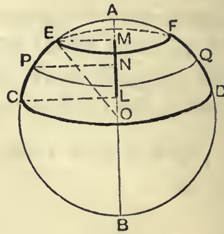
$\therefore$ circumf. of median circle $= \pi(r' + r_1)$.

Also, $h' : r' = h_1 : r_1 = \mu$ say;

$$\therefore \text{ area of frustum} = \text{area of } \mathsf{VAB} - \text{area of } \mathsf{VCD}$$
$$= \tfrac{1}{2}(h'.\,2\pi r' - h_1\,.\,2\pi r_1) = \pi\mu(r'^2 - r_1{}^2)$$
$$= \pi\mu(r' - r_1)(r' + r_1) = \pi(h' - h_1)(r' + r_1)$$
$$= h \times \text{circumf. of median circle.}$$

* A part contained between the base and a parl. plane.

Theorem 26.—(i.) '**The area of the curved surface of a zone of a sphere is the product of its altitude and a great circle; (ii.) the area of a sphere is the product of a diameter and a great circle.**' (**Area** $= 4\pi r^2$.)



If CDEF is a zone of a sphere, centre O, and diam. AB ⊥ parl. planes EF, CD; cut the zone by parl. sections into a number of small zones as EFPQ, and draw perps. CL, PN, EM to AB in the plane ACB.

Then, if section PQ moves to coincidence with EF, the surface of EFPQ becomes a frustum of a rt. circ. cone whose area is 2π EM . EP.

And EP becomes tangt. at E and ⊥ OE; also EM ⊥ MN;

∴ ME : OE = cos MEO = MN : EP;

∴ rect. EM . EP = MN . OE;

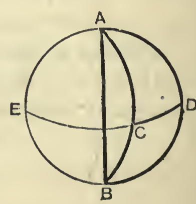
∴ area of zone EFPQ = 2π EM . EP = 2π MN . OE
= MN × circumf. of great circle;

∴ area of CDEF = sum of small zones
= ML × circumf. of great circle
= alt. × great circle.

And area of sphere = sum of areas of zones from A to B
= diam. × great circle.

Ex. Show that the curved surface of a zone of a sphere is equal to that of a right cylinder of equal height, base radius that of the sphere.

Theorem 27.—'**The area of a lune of a sphere is the product of a diameter and the median arc.**'



If ACBD is a lune of a sphere, ECD the great circle perp. to diam. AB; then CD is the median arc.

Hence, if CD = μ . circumf. of ECD,

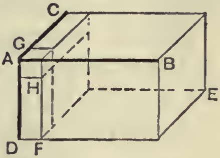
area of lune = μ . area of sphere
= μ . circumf. of ECD × diam.
= CD × diam.
= diam. × median arc.

Theorem 28.—'Two cuboids are congruent which have three concurrent edges of one equal, each to each, to three concurrent edges of the other.'

This needs no demonstration.

Theorem 29.—'The volume of a cuboid is the product of measures of its edges.'

If λ, μ, ν are measures of the edges AB, AC, AD of cuboid AE;
from A along AB make a decimal scale of units, and through the divisions draw planes parl. to ACD;
these form a similar scale of cuboids, unit FC; and the point B and plane BE come between the same divisions of the two scales;
$\therefore$ vol. AE $= \lambda$. FC.

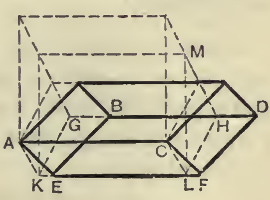


Similarly, making scales of units along AC, AD,
vol. FC $= \mu$FG $= \mu\nu$. GH $= \mu\nu$ unit cubes;
i.e. vol. AE $= \lambda\mu\nu$.

Cor.—'Volume of a cuboid = base × altitude.'

Theorem 30.—'The volume of a paralhedron is the product of base and altitude.'

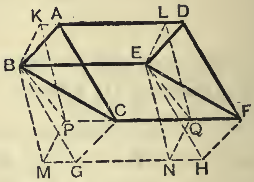
If AB, CD are opp. faces, AEFC a base of paralhedron AD;
through A, C draw parl. planes AKG, CLH perp. to AC, making parn. AH.
Then base AKLC = base AEFC,
and fig. HCLFD $\equiv$ GAKEB;
$\therefore$ parns. AH, AD have equal vol., base, and alt.



Similarly, drawing planes through AC, KL, perp. to base AL, a cuboid AM is formed of same vol., base, and alt. as AH.
$\therefore$ vol. AD = vol. AM = base AL × alt.
= base AEFC × alt.

Cor.—'Paralhedra of equal bases and altitudes are equal.'

Theorem 31.—'The volume of a triangular prism or wedge is the product of base and altitude.'



If ABC, DEF are tr. faces of the wedge or prism AF, complete paralhedron AGDH by planes through BE, CF parl. to AF, AE; and through B, E draw parl. planes BPK, EQL perp. to guide edge BE, forming the paralhedron KMLN.

Then fig. KBPCA ≡ LEQFD,

∴ wedge AF = wedge KQ in vol.;

but rt. wedge KQ ≡ rt. wedge MQ = wedge GF, in vol.;

∴ wedge AF = wedge GF in vol.

= $\frac{1}{2}$ parn. AGDH

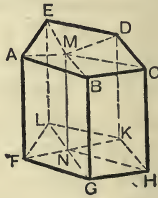
= base ABC × alt.

We have proved incidentally:

Cor.—'A plane diagonal of a paralhedron bisects it; i.e. divides it into wedges of equal volume.'

Note. When the faces of a paralhedron bisected by the diagonal plane ⊥ the other faces, the wedges are evidently congruent, as we have assumed in the above proof.

Theorem 32.—'The volume of a prism is the product of base and altitude.'



If ABC..., FGH... are parl. faces of a prism, and planes are drawn through the guide edges and a parl. line MN; the prism is divided into a number of wedges MFG, MGH, &c.

∴ vol. of prism

= sum of wedges MFG, MGH, &c.

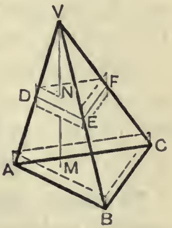
= alt. × sum of bases MAB, MBC, &c.

= alt. × area of base ABCDE.

Ex. 1. Show that the square of diagonal of a cuboid is the sum of squares of its edges.

Ex. 2. Calculate the volume of a prism, alt. 20 ft., base a regular octagon of 1 ft. side.

Theorem 33.—'The volume of a pyramid is one-third of the product of base and altitude.'



If VABC is a pyramid, area of base $ABC = a$, and alt. $VNM = h$;
divide the total altitude VM into any number n of equal parts y, so that $ny = h$;
construct on each section an upper right prism, and on all but the lowest a lower right prism, of alt. y.

Then if S' is the volume of all the upper prisms,
S_1 " " " lower "
and S " " of the pyramid;
$S > S_1$ but $< S'$.

But $S' - S_1 =$ lowest prism $= y \cdot ABC$, which can be made smaller than any given cube, however small, by making y small enough;

$\therefore$ if the points of division move to coincidence, $S' - S_1$, and therefore $S' - S$, becomes zero;

$\therefore$ $S =$ limit of S' when y becomes zero.

Now, the area of any section DEF is proportional to the square of its height $VN = \mu VN^2$, say;

$\therefore$ $a =$ area $ABC = \mu h^2 = \mu n^2 y^2$,

and the volumes of successive prisms from V are

$y \cdot \mu y^2,\ y \cdot \mu(2y)^2,\ y \cdot \mu(3y)^2 \ldots,\ y \cdot \mu(ny)^2.$

$$\therefore\ S' = \mu y^3(1^2 + 2^2 + 3^2 \ldots + n^2) = \mu y^3 \frac{n(n+1)(2n+1)}{6}\,*$$

$$= \frac{\mu n^3 y^3}{3} + \frac{\mu n^2 y^3}{2} + \frac{\mu n y^3}{6} \text{ (multiplying out)}$$

$$= \frac{ha}{3} + \frac{ya}{2} + \frac{\mu h y^2}{6}.$$

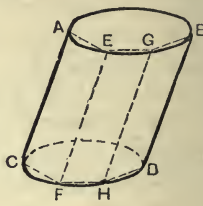
$$\therefore\ S = \frac{ha}{3} \text{ (limit of } S' \text{ when } y \text{ is zero).}$$

Cor.—'Pyramids of equal bases and altitudes are equal.'

Ex. Calculate the volume of a pyramid, alt. 12 cm., base a regular hexagon of 2 cm. side; and the volume of a frustum of this pyramid of height 8 cm. (See p. 271.)

* By a well-known algebraic formula.

Theorem 34.—'The volume of a cylinder is the product of base and altitude.'



If **ABCD** is a cylinder, and **AC, BD, EF,** &c. are guide edges;
then vol. of prism **AEG...CFH** = base × alt.

Hence, if the guide edges **AC, EF...** move to coincidence along the surface,
vol. of cylinder **ABCD** = base × alt.

Theorem 35.—'The volume of a cone is one-third of the product of base and altitude.'

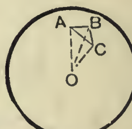
If **VAB** is a cone, and **VA, VC...** vertical edges;
then vol. of pyramid **VAC...** = $\frac{1}{3}$ base × alt.

Hence, if the vertical edges **VA, VC...** move to coincidence along the surface,
vol. of cone **VAB** = $\frac{1}{3}$ base × alt.

Cor.—'A cone is one-third of the cylinder of the same base and altitude.'

Note. If r is the radius of base, h the altitude of a right circular cone or cylinder, the volume is $\pi r^2h/3$ or $\pi r^2 h$.

Theorem 36.—'The volume of a sphere is that of a pyramid whose base is the surface of the sphere, and altitude the radius.'



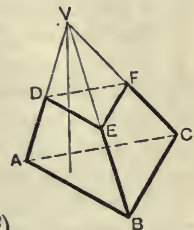
If the whole surface of a sphere, centre **O**, is divided into small triangles **ABC**,
the volume of the pyramid **OABC** on the plane base **ABC** is $\frac{1}{3}$ base × alt.

Hence, if the points of the bases move up to coincidence along the sphere, the sum of all bases becomes the surface, and of all pyramids the volume, of the sphere, and the altitude becomes the radius;
∴ vol. of sphere = $\frac{1}{3}$ surface of sphere × radius.

Cor.—'The volume of a portion of a sphere cut out by a cone whose vertex is the centre is $\frac{1}{3}$ surface of portion × radius.'

Theorem 37.—'The volume of a frustum of a pyramid or cone is that of a pyramid whose altitude is that of the frustum, and base the sum of the two bases and their mean proportional.'

If ABC...DEF... is a frustum of pyramid or cone, vertex V, and the areas of ABC, DEF are a', a_1, the alts. " " " " h', h_1, and alt. of the frustum h;



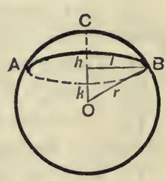
then if $a' = \mu h'^2$, $a_1 = \mu h_1^2$,
mean prop. of a', $a_1 = \sqrt{a'a_1} = \mu h' h_1$;
$\therefore$ vol. of frustum AF $= \frac{1}{3}\mu h'^3 - \frac{1}{3}\mu h_1^3$

$$= \frac{\mu}{3}(h' - h_1)(h'^2 + h'h_1 + h_1^2)$$

$$= \frac{h}{3}(a' + a_1 + \sqrt{a'a_1}).$$

Theorem 38.—'The volume of a segmental cap,* altitude h, of a sphere of radius r, is $\pi r h^2 - \dfrac{\pi h^3}{3}$.'

If ACB is a segmental cap of a sphere, centre O; then OAB is a cone.



If k is alt. of cone, l the radius of plane section AB—i.e. of base of cone—then
$l^2 = r^2 - k^2 = h(2r - h)$.

Also, vol. of solid sector OABC

$$= \frac{1}{3} \text{ area ACB} \times \text{rad. of sphere}$$

$$= \frac{1}{3}h \times 2\pi r^2 = \frac{2\pi r^2 h}{3}.$$

And vol. of cone OAB $= \dfrac{\pi l^2 k}{3} = \dfrac{\pi h(2r - h)(r - h)}{3}$;

$\therefore$ vol. of cap $= \pi r h^2 - \dfrac{\pi h^3}{3}$.

Cor.—'The volume of a zone can be found as the difference of two segmental caps.'

Ex. Calculate the volume of a zone, alt. 2 cm., dist. of greater face from centre of sphere 1 cm., rad. of sphere 4 cm.

* A cap is a portion cut off by a plane surface.

EXAMPLES—XLVIII.

PLANES AND SOLID FIGURES.

1. The angle of two planes is that of two perpendiculars to the planes.

2. A plane is symmetrical in space about any perpendicular.

3. Find the locus of points in space equidistant from two fixed points.

4. If **AP** is perpendicular to the plane of an isosceles triangle **ABC** (**AB** = **AC**), **BC** is perpendicular to the plane through **AP** bisecting **BC**.

5. Find the locus of points equidistant from all points of a circle.

6. A right triangle revolves about a side of the right angle. Show that the opposite vertex describes a circle.

7. If the distances **PA**, **PB** of a point **P** from fixed points **A**, **B** are given in magnitude, **P** lies on a fixed circle.

8. Find the locus of points at a given distance from a given plane.

9. The angle formed by two planes on a plane perpendicular to one of them is greatest when the third plane is perpendicular to both.

10. The locus of points in a plane at a given distance from a given outside point is a circle.

11. A plane perpendicular to any line is perpendicular to any plane through the line.

12. Through a given line in a plane draw a perpendicular plane.

13. Two planes which meet form equal angles on any two parallel planes cutting them.

14. Through a point draw a line parallel to a given line in a plane.

15. Through a point draw a plane parallel to a given plane.

16. The locus of points at a given distance a from a given point, and at a given distance b from a given plane, is a circle.

17. Find the locus of points dividing in a given ratio (i.) all lines from a given point to a given plane, (ii.) all lines bounded by two planes.

18. Construct the common perpendicular of two non-intersecting non-parallel straight lines.

19. The common perpendicular of two non-intersecting lines is the least line that can be drawn from one to the other.

20. All points dividing in a given ratio a line bounded by two straight lines lie in a certain plane.

21. No two straight lines joining points in two non-intersecting non-parallel lines are in the same plane.

22. If two faces of a trihedral angle slide along fixed planes, the third face is always parallel to a fixed plane.

23. The dihedral angle of two planes is proportional to their angle.

24. Two planes which meet cross one another and form four angles.

25. If two planes cross, opposite dihedral angles are equal.

26. Any three faces of a tetrahedron are together greater in area than the fourth.

27. Find the locus of a point whose distances from two fixed planes (i.) are equal, (ii.) have a fixed ratio.

28. If the joins of a point to the contour of a given figure are multiplied by a given ratio, their ends form the contour of a similar and similarly situated figure.

29. Find the locus of a point dividing in a given ratio the joins of a given point to points on a sphere.

30. If a sector of a circle is revolved about its bisector of angle, a cube can be inscribed in the resulting solid sector.

31. The points of contact of tangents from an outside point to a sphere are in a plane and on a circle. (Rotate circle about diam.)

32. The locus of points from which equal tangents can be drawn to two spheres is a plane. (Radical plane.)

33. The radical planes of three spheres meet in a straight line.

34. The centroids of parallel plane sections of a trihedral angle lie on a line through its vertex.

35. The joins of the vertices of a tetrahedron to the centroids of opposite faces are concurrent, and divide each other in the ratio 1 : 3.

36. Find the locus of a point equidistant from three given points.

37. Circumscribe a sphere to a tetrahedron.

38. Find the locus of a point equidistant from the faces of a trihedral angle.

39. Inscribe a sphere to a tetrahedron. How many e-spheres are there?

40. Inscribe a sphere in a right circular cone.

41. The vertex of a right conical surface divides the line of centres of two inscribed spheres in the ratio of the radii.

42. The lengths of all common tangents of the spheres of Ex. 41 whose directions pass through the vertex are equal.

43. Circles of a sphere equally distant from its centre are equal.

44. Calculate the following surfaces: (i.) cube, edge 1·6″; (ii.) regular tetrahedron, edge 1″; (iii.) right circular cone, height 6 cm., base radius 2·5 cm.; (iv.) right circular cylinder, height 3·2″, base radius 1·8″.

45. A zone is cut out of a sphere, radius 5 cm., by two planes distant 3·2 cm., 4·8 cm. from the centre; calculate its area.

46. Calculate the ratio to the whole surface of the earth of the part contained between two meridians of longitude 23° 30′ and 57° 50′ W.

47. What is the area of the cap of the Arctic Circle of the earth? (Earth's radius, 3960 miles; arc from pole to edge of circle, $23\frac{1}{2}$°.)

48. A heap of stones has the form of a frustum of a pyramid of square base; side of lower base 30 ft., of upper base 27 ft., height $2\frac{1}{2}$ ft.; find the height of the vertex, and the volume of the heap.

49. If a cubic foot of the heap in Ex. 48 weighs 168 lb., what is the total weight in tons?

50. If the volume in Ex. 48 is calculated as a cuboid of the same height, with the square half-way up as base, what is the error?

51. A wedge can be divided into three pyramids of equal volume.

52. A peat-stack runs from a rectangular base 10 ft. by 30 ft. to a horizontal line at the top 10 ft. high. If the ends slope at the same angle as the sides, calculate its volume. (Divide it into a pyramid and a wedge.)

53. A pipe, average diameter 2 ft., brings water from a lake 70 miles distant; how much water is there in the pipe at a given moment?

54. A reservoir is $\frac{1}{2}$ mile long, $\frac{1}{4}$ mile broad; if 10,000,000 gallons are drawn off in a day, how long will it take to lower the level 15 ft.? (Take 1 gal $= \cdot 16$ cub. ft.)

55. The locus of a point whose distances from two points $4''$ apart are in the ratio $3:2$ is a sphere. (Find locus in a plane and rotate.)

56. Find the volume of the sphere in Ex. 55.

57. A funnel has an upper diameter $6''$, lower diameter $1''$, depth of conical part $4''$, length of cylindrical part $4''$. What surface has it?

58. A bucket has the shape of a frustum of a cone; upper diameter 1 ft., lower $10''$, depth 1 ft. Calculate its volume.

59. A stone of irregular shape is put into a cylinder, $10''$ diameter, containing water, and causes the water to rise $6''$. Find the volume of the stone.

60. The base of a cylindrical can is an ellipse, axes $12''$ and $6''$, and the height $10''$. How many gallons does it hold? (Area of ellipse $= \pi \times$ product of semi-axes; 1 cub. ft. $= 6 \cdot 23$ gals.)

61. Calculate the area of the right projection of a circle of radius 5 cm. upon a plane at an angle of $50°$ to its plane.

62. An oak pillar 12 ft. high has as base a regular hexagon of 1 ft. side. Calculate its surface and volume.

63. If the pillar of Ex. 62 was cut out of a trunk of 3 ft. diameter and the same length, how much was cut away?

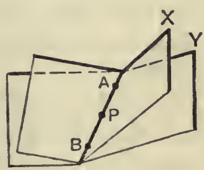
64. A tree stands 100 ft. high on a base of 17 ft. diameter. Taking it as a cone, how many tons of wood does its trunk yield? (Take 1 cub. ft. to weigh 50 lb.)

65. Two caps $3''$ and $4''$ deep are cut off a sphere of $12''$ diameter. Calculate the volume and surface of the remainder.

Now that some familiarity with the figures of two planes has been obtained, we can show that it is possible to give **formal** proof (i.e. directly from the definitions of plane and straight line) of Thh. 2, 3, and 6 of Chapter I., which we there derived experimentally.

Theorem 2, Ch. I.—'Two straight lines coincide entirely when two points of each coincide.'

The folds of two planes may be placed so as to have two points A, B common;



one side of each plane may be kept fixed, and their other sides X, Y moved about their respective folds AB.

As these folds are the boundaries of the non-moving part of the planes, neither fold moves when X, Y move;

if then Y is moved until some point P of the fold of X is on Y, and the two parts X, Y are then rigidly fastened so as to move together,

P, being on the fold of X, does not move, and therefore it is on the fold of Y, since all points of Y move which are not on the fold—i.e. every point P of the fold of X is on the fold of Y.

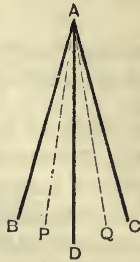
It should be noticed, however, that we here make an **assumption** not generally necessary for plane geometry, but necessary for the geometry of space, and which has already been given on the first page of this chapter—viz. that if a plane turns completely round the line joining two points on it, it sweeps out the whole of space. It was because of the difficulty of this notion that this formal proof had to be deferred.

It is impossible, of course, to prove this experimentally or formally; its justification is that it is consistent with all those results of geometry which we can test, and that it enables us to treat consistently and successfully such problems as movements of stars which are some billions of miles away. It is assumed at some stage or other in all systems of space geometry.

We can now also establish **formally** Theorems 3 and 5 of Ch. I.

Theorems 3, 5 of Ch. I.—'Every straight line has a mid point, and every angle a bisector, about which it can be reversed.'

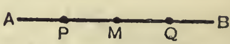
If **BAC** is any angle, we may suppose two lines **AP, AQ** to turn in the plane from the sides **AB, AC** at the same rate—i.e. so that at any instant **CAQ** can **reverse** on to **BAP**.



These moving lines meet in some line **AD** in the angle;

∴ the angle **DAC** reverses on to **DAB**, similarly **DAB** on to **DAC**; and the whole angle **BAC** reverses about the bisector **AD**.

Similarly, if points **P, Q** move from **A, B** along the line **AB** in opposite directions at the same rate—i.e. so that at any instant **QB** can **reverse** on to **AP**—



these moving points meet in a point **M**, the mid point of **AB**, about which the line can be reversed.

The above principle of reversibility of an angle and of a straight line is very important, and has been entirely overlooked in our text-books. Without it, for example, it is impossible to establish the congruence of two triangles which have two sides and their angle of each equal, or two angles and corresponding side equal, **when the triangles have contrary aspects.**

The difficulty will be readily understood by substituting two spherical triangles (formed by arcs of great circles on a sphere) of contrary aspect, and trying to establish their congruence.

As the fundamental property of an isosceles triangle is either proved by direct turning over, or by the congruence of two triangles of **contrary aspect**—e.g. those formed by bisector of vertical angle—it is evident that this property cannot be proved without the assumption of this principle of reversibility, which therefore ought to be explicitly given.

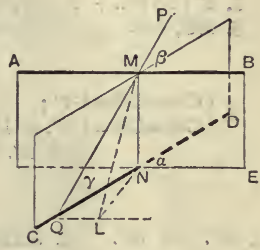
Note. By turning over **DAC** on to **DAB**, and then reversing it on itself in this position, it is readily seen that **CAD** can be **rotated** into coincidence with **DAB**.

We can now also justify our restriction of the lines to one plane, in the definition of lines in the same direction.

'Two non-coplanar lines cannot have the same direction;'* or,

'Two lines in the same direction are coplanar, and therefore parallel.'

If AB, CD are non-coplanar lines, MN their common perp.; CED a plane through CD parl. to AB, and cutting plane AMN in NE; then DNE = ang. of planes $= \alpha$ say.



Draw any transversal PMQ of AB, CD, make QL parl. to AB and NE; plane MNL perp. to QL.

Then if angs. PMB, MQN made by transversal PQ with AB, CD are β, γ;

$$\cos\beta = \cos \mathrm{MQL} = \frac{\mathrm{QL}}{\mathrm{QM}} = \frac{\mathrm{QL}}{\mathrm{QN}} \cdot \frac{\mathrm{QN}}{\mathrm{QM}}$$

$$= \cos \mathrm{NQL} \,.\, \cos \mathrm{NQM} = \cos \mathrm{DNE} \,.\, \cos\gamma$$

$$= \cos\alpha \,.\, \cos\gamma.$$

$\therefore$ unless $\cos\beta$ and $\cos\gamma$ are each zero—i.e. unless $\beta = \gamma = 90°$—(in which case PQ is MN and not **any** line), we can only have $\cos\beta = \cos\gamma$ when $\cos\alpha = 1$;

i.e. $\beta = \gamma$ when only $\alpha = 0$—i.e. when only the planes AMN, CMN coincide, and CD, AB are coplanar.

Thus two non-coplanar lines make equal angles towards the same parts with their common perpendicular only; and hence two lines having the same direction must be coplanar—i.e. parallel.

* That is, cannot make equal angles towards the same parts with any third line whatever which meets them.

ANSWERS TO EXAMPLES.

I.

1. 4″; $2\frac{1}{2}$ cm. + . 2. 60°. 4. 60°. 5. $2\frac{1}{2}$″.
6. AC, AD. 7. 45°, 135°, 90°. 9. 40°, 140° 14. 3 cm.

II.

2. $\begin{cases} CA = CB = 1{\cdot}4'' ; \\ DA = DB = 1{\cdot}8'', \text{ or } 1\frac{7}{8}'' - . \end{cases}$ 4. 73°, 107°. 5. 45°, 135°.
6. 2″. 7. 129°, 51°, 129°.

III.

3. 90°. 4. 60°. 5. Altitude from base, 3·1 cm. 6. QR = 1·9″, or ·9″.

IV.

1. Alt. from base, $2\frac{1}{4}$ cm. 3. 70°; base, 1″. 4. 60°. 5. 41°.
6. 4·8 cm. 7. 3 cm. 8. 14°.

V.

1. 2·8 cm., 5·3 cm. 3. 33°; long. diag. 2·1″. 4. 360°; long. diag. 5·6 cm.
6. ·87″, or $\frac{7}{8}$″. 7. 4 cm. 8. R = ·77″, or $\frac{3}{4}$″. 9. R = 2·5 cm.

VI.

1. 61°, 30°, 22°, $14\frac{1}{2}$°. 2. BP = 3·5 cm. 5. AP = 1·31″. 6. 2·8 cm.

VII.

2. Fourth side, 1·82 mi. 3. 1/50,000. 4. 7/288.
5. Unit, 1″. 6. Unit, ·72″. 7. 1/190,080; $1\frac{7}{8}$″.
8. 1/52,800. 9. $22\frac{1}{2}$ mi. 11. 3/100,000.

XI.

2. 1·45″, or $1\frac{7}{16}$″. 3. 2·39″, or $2\frac{3}{8}$″. 4. Alt. = 1·52″, or $1\frac{1}{2}$″.
5. 1″; 60°. 6. 1·08″, or 1·1″. 7. 1·72″, or $1\frac{3}{4}$″.

XII.

1. $1\frac{1}{4}$″. 2. ·755″, or $\frac{3}{4}$″. 3. 1·9″, or $1\frac{7}{8}$″. 4. 1·77″, or $1\frac{3}{4}$″.

XV.

1. 1″; ·77″, or ¾″.   2. ½″.

XVI.

2. 3 sq. in.   3. ·974 sq. in.   4. 1·75 sq. in., 3·5 sq. in.; $1\frac{9}{16}$ or 1·56 sq. in.

XVII.

1. 36·64 sq. cm.   2. Lengths: 8·167″, 16·96 cm., 34·56 yd.; Areas: 53·09 sq. in., 22·91 sq. cm., 95·06 sq. yd.

3. Arcs: 1·885, 2·513, 3·77 cm.; Areas: 2·262, 3·016, 4·524 sq. cm.   5. 459.

XVIII.

1. 80° 24′.   2. 1·46″.   3. 1·23″.   4. 48 yd.

5. 3 ft. 6 in.   6. 35·75 ft.   10. 5145 yd.   11. 104 ft.

12. 18 mi.; parm.   13. 2 mi. E. and W.   14. 47° 10′.   15. 4·76 cm.

16. Alt. 1·15 cm.   17. Short diag. 1·69″.   18. Ang. 70° 43′.   22. 3·53 cm.

23. 1·98″.   24. 4·95 cm.   27. 17·32 mi.   31. Perp. 2·44 and 3·35 cm.

32. CD=3·6 cm.   33. 3¼, 6½ cm.   34. 3·96 cm.   35. 3·35, 4·3 cm.

36. 1¼″.   37. 1·758″.   39. BD=1·6″.   40. EF=1¼″.

41. Other diag. 2·34″.   42. Long diag. 7·27 cm.   44. 9 : 49.

46. 3·91 sq. in., 10428 sq. yd., 472·5 sq. ft.   47. 2992 sq. yd.; diag. 83·14 yd.

48. 4·74 cm.; 4·37″; 5 mi.; 346 yd.

49. $\sqrt{3}$″, or 1·732″; ·866″; 2·598 cm.; 4·33 ft.; 6·062 yd.

50. 400 yd.   51. 116·6 ft.   52. 9.   53. 9 : 25.

54. 18·85 ft.; 87·96″, or 7 ft. 4″; 11 ft.   55. 280, 720, 480.

56. 28·27 ft., 616 sq. in., 9·625 sq. ft.   57. 1257 sq. ft.

58. ·262, ·393, ·524, ·628, 1·05 sq. in.   59. 2, 1·14 sq. in.   60. 32 sq. cm.

61. 36 sq. cm.   62. 7·73 sq. cm.   63. 1·732″, 1·5″, 1·3 sq. in.

XXI.

1. 10, 12, 16 rt. angs.; 4 rt. angs.   2. 108°, 120°, $128\frac{4}{7}$°.   3. c=·804″.

4. c=·88″.   5. 60°; c=2·77 cm.   6. AD=·72″.   7. 1¾″.

11. Alt. 3·2 cm.   13. 120°, 30°; b=1·5″.   14. 2·12 cm.

15. c=4·23 cm.   17. a=1·92″.   18. b=2·59 cm.   19. c=2·16″.

20. Side, 4·41 cm.   21. Diag. 4·67 cm.   26. Sides, 1·84″, ·79″.

27. Sides, 3·21 cm., 2·09 cm.   28. Sides, ·74″, 1·14″, 1·13″.

29. Other diag. 2·39″.   40. 7·43 mi.   42. Chd. 1·54″.

43. Sides, 4·9, 3·37 cm.   44. 8·09, 5·27 mi.   45. 4·73 mi.

47. Side, 2″.   48. Side, 3·97 cm.   49. Side, 2·35″.   51. Diag. 3·3″.

52. 1″, 1·73″.   54. b=5·55 cm.   56. BC, 3 cm.; BD, 4·36 cm.

57. 72°, 36°.   58. 5·5 cm.   59. CD, AD, 1″; 108°, 36°.   60. 108°.

XXIV.

6. 26° 34′. **8.** 78·2 ft. **9.** 50 ft. **11.** AC, 5·43 cm. **12.** 3·14″, 2·66″.
14. 5·05″. **19.** Line divides angle into 35° 24′ and 27° 36′.
20. Short diag. 2·28 cm. **21.** Side, 2·35 cm. **25.** Middle side, ·94″.
26. Side, 2·4 cm. **27.** 2·65. **28.** 1″. **29.** DC = ·28″.
30. 1·8″, ·8″, 1·2″. **32.** CD, 9 cm. **33.** c = 1·5 cm.
34. 1 : 9. **35.** Greatest length, 5·6″. **36.** ·618″. **37.** $\frac{\sqrt{5}+1}{2}$, or 1·618.
38. 4 : 1. **39.** AB, 1·62″. **41.** Rad. 1·44 cm. **42.** Rad. 2·16 cm.
43. b = 1·91 cm. **46.** 4·32 sq. cm., 3·12 sq. cm., 214·63 sq. cm.
47. 2·08 cm., 1·77 cm., 14·64 cm. **48.** 16·24 sq. cm.
50. Side, 3·08 cm. and 1·73 cm. **51.** 164·5 sq. mi.
52. ·24 sq. mi., or 153·6 acres. **53.** 4·43 sq. cm., 8·36 sq. cm., ·5 sq. in.
54. Side, 3·46 cm. **55.** Side, 2·2″. **56.** Side, 3″.
57. Parts, 1·661″, ·339″. **58.** ·865″. **59.** 1·94″.
60. Side, 2·44 cm. **61.** Side, 1·32″. **63.** 3·45″. **64.** Third side, 1·87″.
65. Alt. from diag. to rt. ang. ·6″. **66.** Side, 5·4 cm. **67.** PD = ·87″.
68. AC, 2·5 cm.; AD, 10 cm.; AP, 6·25 cm. **71.** 4″. **72.** 11·1 cm.

XXVII.

5. 1·62″. **7.** Central dist. of chd. 1·51″. **9.** Rad. 1·04″. **14.** 3·96″.
15. 2 cm. **18.** Dist. 5·13 cm. **20.** ⅔″, or ·67″.

Page 119.

1. 8100 c. in., 6000 c. ft. **2.** 343 c. in., 2197 c. ft., 13·824 c. c.

Page 120.

1. 11·5 c. c. **2.** 4·524 sq. cm., 27·14 c. c.

Page 121.

1. 12·15 c. in. **2.** 47·12 sq. cm., 37·7 c. c.

Page 122.

1. 50·29 sq. cm., 32·17 sq. in., 314·2 sq. ft.;
33·5 c. c., 17·15 c. in., 524 c. ft.
2. 24,883·2 mi. **3.** 24,881·4 mi., 197,060,800 sq. mi.

XXVIII.

1. 5184 c. ft. **2.** 9504 c. in. **3.** 6375 c. in. **4.** 84,480 c. yd.
5. 36 c. in. **6.** 216 c. in. **7.** 1357. **8.** 4·9.
9. 330 sq. in. **10.** 3,440,853 c. yd. **11.** 25·13 c. in.
12. 268·1 c. yd., 207·25 sq. yd. **13.** 4·189 c. ft. **14.** 2·094 c. ft.
15. 3″. **16.** 265. **18.** 25,943 sq. yd.

PAGES 130, 131.

$\triangle$, 9·8 sq. cm.; r, 1·23; r_a, 2·45; r_b, 3·27; r_c, 9·8; R, 3·57; OI, 2 cm.

PAGE 132.

Lengths: 18·22, 1·619, 3·645, 11·39, 15·39.
Areas: 26·42, 2·348, 5·284, 16·51, 22·31.

XXX.

1. Mean part, 2″. 4. $\frac{\sqrt{3}}{2}$, 1·3 sq. in., 12 sq. cm. 5. 2·12 sq. in.
6. $\triangle$, 94·1 sq. ft.; R, 9·18; r, 4·09; r_a, 8·56; r_b, 13·44; r_c, 18·82 ft.
9. 2·1 cm. 10. Dist. from cent. 2·1, 2·9 cm. 11. $1\frac{3}{8}$″, $1\frac{1}{8}$″ from ends.
13. 1 cm., 5 cm. from ends. 15. 3·873, 38·73 mi. 16. 1·94″.
18. New side, 1·414″. 19. Side, ·62″. 20. Line cuts CB 1·5″ from C.
22. Side, 5·265 cm. 26. Line cuts AC ·409″ from A.
27. 1·57 cm. from A; a. 28. 2·598, 10·392 sq. cm.
29. 8 : $3\sqrt{3}$, or 1·539. 30. Chds. 1·27 rad., 1·55 rad.
31. 3·61 sq. cm. 38. Side of sq. 7·65 cm.

XXXIV.

5. Diag. 5·23 cm. 8. 34°, 97°, 34°.
9. Short diag. 2·98 cm. 16. A, 135°; D, 60°; E, 105°; EC, 3″.
17. 80°, 39°; b, 5·72″. 19. b, 2·61 cm. 22. Alt. from BC, 1·53″.
23. a, 8·02 cm. 28. Side, 1·5″. 31. 2·13″.
34. Diag. 3·62″. 35. Longest side, 2·55 cm. 49. b, 4·11 cm.

PAGE 163.

4. 2·789, 1·395. 5. ·968.

PAGE 164.

3. 4·24 cm. 4. 86·6 ft.

PAGE 166.

2. ·578, 1·41; ·922, 2·38; ·8035, ·595.

XXXV.

2. (i.) $3\frac{1}{4}$; (ii.) $2\sqrt{2}$, or 2·828; (iii.) 1.
3. (i.) $\frac{1+3\sqrt{3}}{2}$, or 3·098; (ii.) $1+\sqrt{2}$, or 2·414; (iii.) $\frac{3+5\sqrt{3}}{6}$, or 1·943.
6. (i.) $\frac{1+\sqrt{3}}{2\sqrt{2}}$, or ·966; (ii.) 1; (iii.) $\frac{1+\sqrt{3}}{2\sqrt{2}}$.

7. (i.) 45°; (ii.) 60°; (iii.) 30°; (iv.) 0, or 60°; (v.) 60°.

8. (i.) $-\frac{b}{\sqrt{a^2+b^2}}, \frac{a}{b}$; (ii.) $-\frac{x}{\sqrt{1+x^2}}, \frac{1}{x}$; (iii.) $\frac{p-q}{\sqrt{2(p^2+q^2)}}, \frac{p+q}{\sqrt{2(p^2+q^2)}}$

11. $\frac{1}{2}, -\frac{1}{\sqrt{2}}, -\sqrt{3}, -\frac{1}{\sqrt{2}}, \frac{1}{2}, -\frac{1}{\sqrt{3}}$.

12. (i.) $kl+\sqrt{(1-k^2)(1-l^2)}$; (ii.) $-\sqrt{1-k^2}, -\sqrt{1-l^2}, -\sqrt{1-l^2}/l, -\sqrt{1-k^2}$.

13. 70·02 ft. 14. 746·4 ft. 15. 46·5 ft. 16. 3° 49′, 176 ft.

17. 662 ft. 18. 16·66 yd., or 49·96 yd. 19. 4·276 mi. 20. 8° 28′.

Page 170.

4. (i.) C, 55°; b, 668·2; c, 566·7; R, 345·9 yd.
(ii.) B, 45°; a, 1·38; c, 1·027; R, 6405 cm.

Page 171.

1. (i.) A, 41° 48′; B, 78° 34′. (ii.) A, 44° 25′; B, 57° 7′.
(iii.) A, 35° 49′; B, 48° 28′.

Page 172.

2. (i.) Δ, 89·3; r, 3·97; r_b, 11·9; R, 9·07.
(ii.) Δ, 228·6; r, 6·351; r_b, 20·78; R, 14·43.

XXXVI.

1. 81° 51′, or 98° 9′. 2. 57° 58′, or 122° 2′. 3. 46° 50′.
4. B, 44° 25′; C, 34° 3′. 5. A, 22° 20′; C, 108° 13′. 6. 3·16″.
7. 11·14 ft. 8. 185·4 yd. 9. 168·8 yd. 10. 565.
11. 7335. 12. c, 550·8; a, 314·9. 13. 34° 38′. 14. 41° 48′.
15. C, 85° 40′; A, 52° 37′. 16. 76° 44′. 17. 73° 24′. 18. 46° 59′; 4095.
19. 54° 4′, or 125° 56′. 20. 42° 24′, or 137° 36′.
21. A, 56° 19′; B, 77° 58′; b, 1126 : or A, 123° 41′; B, 10° 36′; b, 211·8.
22. 95° 44′. 23. 281·5, 125·9. 24. r, 39·8, 7·215; r_a, 84·41, 32·22.
25. 72,100, 57,350. 26. $x\sqrt{2}/2$. 27. $2:\sqrt{7}$, or ·7703. 28. $(2{\cdot}1\pm\sqrt{{\cdot}41})/2$.
29. 1·376″, ·7237; 103° 42′, 31° 8′. 30. Side, 8·09 cm.; angs. 72°, 72°, 36°.

XXXVII.

1. 487 or 413 yd. 2. 40·14 ft. 3. 4897 ft. 4. 12,000 ft.

XXXVIII.

1. $\frac{\sqrt{3}-1}{2\sqrt{2}}, \frac{1-\sqrt{3}}{2\sqrt{2}}, \frac{1-\sqrt{3}}{2\sqrt{2}}, \frac{1+\sqrt{3}}{2\sqrt{2}}$. 5. $\sqrt{3}-2, -\sqrt{3}-2$.

XXXIX.

1. 2 sin 66° cos 6°; $\frac{1}{2}$(cos 12° − cos 132°). 2. 96°.

PAGE 180.

1. $\frac{7}{25}, \frac{24}{25}; \frac{1}{49}$, 48·99.

XL.

1. 120°. **2.** 30°. **3.** 0, 180°, 8° 25′. **4.** 66° 8′, or 91° 41′. **5.** 63° 26′. **6.** $-90°$, or 270°; 61° 56′.

PAGE 182.

1760 ft. 847,066 mi.

XLI.

1. $\frac{\pi}{2}, \frac{5\pi}{12}, \frac{3\pi}{5}, \frac{3\pi}{4}$. **2.** 120°, 45°, 144°, 54°. **3.** 34·56 mi.

XLII.

1. c, b. **3.** b. **8.** ·8506″. **9.** 2·598 cm. **10.** $a/2 \sin\frac{\pi}{n}$, $a/2 \tan\frac{\pi}{n}$.
11. $\frac{\sqrt{5}-1}{4}$, or ·309. **12.** 1·464 ft. **13.** $2r\sin\frac{a}{2}$, $r\cos\frac{a}{2}$.
16. 46° 53′, 54° 33′, 58° 19′. **19.** ·9616, ·2747. **20.** ·7454, 1·118.
26. 3°, 48°, 64°. **27.** $-$·9397, $-$·766, 5·671. **28.** 25° 55′, 48° 1′, 16° 15′.
29. 30°; $\pm 90°$, or 60°; $\pm 60°$, or $\pm 30°$. **39.** 60° 57′. **40.** 10·8 ft., 30° 58′.
41. 98·57 ft. **42.** 1980 mi., 12440 mi. **43.** Side, 141·4 ft.; diag. 200 ft.
44. 41° 49′. **45.** 38° 11′. **50.** ·1749″, ·6527″, ·2404 sq. in.
51. AD, 1·368″; CD, 1·009″. **54.** B, 75°. **55.** a, 207·6; b, 260·4 yd.
56. 3302 m., 1,756,000 sq. m. **57.** 67° 18′, or 112° 42′; 236·6 or 72·2 yd.
58. A, 62°; B, 57° 42′; C, 104° 5′; D, 136° 13′.
59. 91° 2′; AD, 601·4; BC, 751·4; DC, 332·8 yd.
60. 1093 m. **66.** Cot 2θ; $-\tan(\alpha-\beta)$. **69.** $2\tan 2\alpha$. **70.** 0.
73. 11,041 ft., 58°. **74.** 3·903 or 3·223 mi.
75. (i.) 0°, or 180°; (ii.) 30°, or 150°; (iii.) 90°; (iv.) 135°, 0°, 90°.
76. $\frac{6+\sqrt{35}}{12}$, or ·9931; $\frac{\sqrt{5}}{4}+\frac{\sqrt{7}}{6}$, or 1 (nearly). **83.** $-63/65$. **84.** $\frac{\sqrt{5}-1}{2}$.
87. (i.) 0, $22\frac{1}{2}°$, 20°. (ii.) 30°, 60°. (iii.) 0, 90°, 180°. (iv.) 45°, 135°. (v.) α, $90°-\alpha$. (vi.) θ, 0, or π; ϕ, 0, or π; θ, 34° 3′; ϕ, 101° 32′. (vii.) θ, 64° 39′; ϕ, 37° 3′.
92. 5·477 mi., 17·32 mi., 54·77 mi. **93.** 69·12 mi. **94.** 1·02″.
95. 95° 33′. **96.** $\frac{\pi}{6}, \frac{\pi}{4}, \frac{5\pi}{6}, \frac{5\pi}{4}$; 72°, 70°, 30°, 40°. **99.** 57·29 cm.
101. (i.) $\frac{\pi}{12}, \frac{13\pi}{12}, \frac{25\pi}{12}\ldots, \frac{5\pi}{12}, \frac{17\pi}{12}, \frac{29\pi}{12}\ldots$. (ii.) 71° 30′, $n.180°+71°\ 30′$. (iii.) $\frac{\pi}{6}$, $2n\pi+\frac{\pi}{6}$.

XLIII.

24. Length of tangt., 1·3″. 25. Side, 1·65″. 26. 1·17″, 1·06″. 29. ·615″. 31. $\frac{1}{4}$″.

Page 268.

2. 96·57 c. ft.

Page 269.

6·93 c. c., 6·67 c. c.

Page 271.

7·33 c. c.

XLVIII.

44. 15·36, 1·732, 70·68, 56·55 sq. in.
45. Curved surf., 50·266, 251·33 sq. cm.; total area, 104·3, 305·4 sq. cm.
46. 103 : 540 (or 1080). 47. 8,172,000 sq. mi. 48. 25 ft., 2032·5 c. ft.
49. 152·4 tons. 50. 1·875 c. ft. (<·1 per cent.). 52. 1333$\frac{1}{3}$ c. ft.
53. 171,600 c. yd. 54. 32·67 days. 56. 24·43 c. in. 57. 64·44 sq. in.
58. ·662 c. ft., or 1144 c. in. 59. 471·2 c. in. 60. 2·04.
61. 50·48 sq. cm. 62. 72 or 77·2 sq. ft.; 31·18 c. ft. 63. 53·65 c. ft.
64. 169. 65. 528·8 c. in.; curved surf., 188·5 sq. in.; total, 373·85 sq. in.

THE END.

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